REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION

INTERNATIONAL OLYMPIAD
"TUYMAADA-2016"
(mathematics)
Second day

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The booklet contains the problems of XXII International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by S. L. Berlov, A. S. Chukhnov, V. L. Dolnikov A. S. Golovanov, L. A. Emelyanov, K. P. Kokhas, F. V. Petrov, A. I. Khabrov, Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

5. The ratio of prime numbers p and q does not exceed 2 ($p \neq q$). Prove that there are two consecutive positive integers such that the largest prime divisor of one of them is p and that of the other is q.

(A. Golovanov)

6. The numbers a, b, c, d satisfy $0 < a \leqslant b \leqslant d \leqslant c$ and $a + c = b + d$. Prove that for every internal point P of a segment with length a this segment is a side of a circumscribed quadrilateral with consecutive sides a, b, c, d, such that its incircle contains P.

(L. Emelyanov)

7. For every x, y, $z > \frac{3}{2}$ prove the inequality

$$x^{24} + \sqrt[5]{y^{60} + z^{40}} \geqslant \left(x^4y^3 + \tfrac{1}{3}y^2z^2 + \tfrac{1}{9}x^3z^3\right)^2.$$

(K. Kokhas)

8. A connected graph is given. Prove that its vertices can be coloured blue and green and some of its edges marked so that every two vertices are connected by a path of marked edges, every marked edge connects two vertices of different colour and no two green vertices are connected by an edge of the original graph.

(V. Dolnikov)

Junior League

5. Positive numbers are written in the squares of a 10×10 table. Frogs sit in five squares and cover the numbers in these squares. Kostya found the sum of all visible numbers and got 10. Then each frog jumped to an adjacent square and Kostya's sum changed to 10^2. Then the frogs jumped again, and the sum changed to 10^3 and so on: every new sum was 10 times greater than the previous one. What maximum sum can Kostya obtain?

(K. Kokhas)

6. Is there a positive integer $N > 10^{20}$ such that all its decimal digits are odd, the numbers of digits 1, 3, 5, 7, 9 in its decimal representation are equal, and it is divisible by each 20-digit number obtained from it by deleting digits? (Neither deleted nor remaining digits must be consecutive.)

(S. Berlov)

7. The numbers a, b, c, d satisfy $0 < a \leqslant b \leqslant d \leqslant c$ and $a + c = b + d$. Prove that for every internal point P of a segment with length a this segment is a side of a circumscribed quadrilateral with consecutive sides a, b, c, d, such that its incircle contains P.

(*L. Emelyanov*)

8. The flights map of air company $K_{r,r}$ presents several cities. Some cities are connected by a direct (two way) flight, the total number of flights is m. One must choose two non-intersecting groups of r cities each so that every city of the first group is connected by a flight with every city of the second group. Prove that number of possible choices does not exceed $2m^r$.

(*D. Conlon*)

SOLUTIONS

Senior League

5. Let $q < p < 2q$. Since p and q are coprime, there are integers x and y such that $px - qy = 1$. The numbers x and y can be changed to $x - qk$ and $y + pk$ for any integer k without changing $px - qy$, so we can assume $|x| < \frac{q}{2}$. The numbers $|px|$ and $|qy|$ are consecutive and positive (they are non-negative and if one of them is 0, then the other is 1, which is impossible since 1 is not divisible neither by p nor by q). Since $|px|$ is less than $\frac{pq}{2}$, its greatest prime divisor is p. On the other hand, $|qy| \leqslant |px| + 1 < \frac{pq}{2} + 1 \leqslant \left(\frac{p+1}{2}\right) q$. Therefore $|y| \leqslant \frac{p+1}{2} \leqslant q$, and every prime divisor of $|qy|$ does not exceed q.

6. Let a, b, c, d be the sides of a circumscribed quadrilateral.

Every point where the inscribed circle touches a side of a circumscribed quadrilateral divides the side into two parts. The parts containing a common vertex of the quadrilateral have equal lengths. Therefore when a tangency point P is chosen on the minimum side, we can found (positive) lengths of all the segments between the tangency points and the ends of the respective sides. Let x_1, x_2, x_3, x_4 be the lengths of these segments, $a = x_1 + x_2$, $b = x_2 + x_3$, $c = x_3 + x_4$, $d = x_4 + x_1$.

Let R — be the inradius of the quadrilateral. Then the angles of the quadrilateral are $2\arctan\frac{R}{x_1}$, $2\arctan\frac{R}{x_2}$, $2\arctan\frac{R}{x_3}$, $2\arctan\frac{R}{x_4}$. For each x_i the function $R \mapsto 2\arctan\frac{R}{x_i}$ is monotonic, continuous and increases from 0 to $\frac{\pi}{2}$ when R is between 0 and $+\infty$. Then for each given x_1, x_2, x_3, x_4 there is an R satisfying the equation

$$2\arctan\frac{R}{x_1} + 2\arctan\frac{R}{x_2} + 2\arctan\frac{R}{x_3} + 2\arctan\frac{R}{x_4} = 2\pi.$$

When the radius is known, it is easy to construct the desired quadrilateral.

7. Let $A = x^4y^3 + \frac{1}{3}y^2z^2 + \frac{1}{9}x^3z^3$. Suppose the inequality is wrong. Then

$$x^{24} < A^2, \qquad \sqrt[5]{y^{60}} < A^2, \qquad \sqrt[5]{z^{40}} < A^2.$$

Or, to put it simply,

$$x < A^{1/12}, \qquad y < A^{1/6}, \qquad z < A^{1/4}.$$

Using these inequalities we find that

$$A = x^4y^3 + \tfrac{1}{3}y^2z^2 + \tfrac{1}{9}x^3z^3 < A^{\frac{4}{12}+\frac{3}{6}} + \tfrac{1}{3}A^{\frac{2}{6}+\frac{2}{4}} + \tfrac{1}{9}A^{\frac{3}{12}+\frac{3}{4}} = \tfrac{4}{3}A^{\frac{5}{6}} + \tfrac{1}{9}A.$$

We immediately get a contradiction if for x, y, $z > \frac{3}{2}$ the right-hand side of the last inequality does not exceed A. This is indeed so: the inequality $\frac{4}{3}A^{\frac{5}{6}} + \frac{1}{9} \leqslant A$ is equivalent to $A \geqslant (\frac{3}{2})^6$, which is true for our x, y, z: $A > x^4y^3 > (\frac{3}{2})^6$.

8. We prove the statement by induction in the number n of vertices. For $n = 1$ it is obvious. Suppose the statement is proved for graphs with $n-1$ vertices. Consider a graph G with n vertices and remove one its vertex v so that the graph $G_1 = G \setminus v$ is connected. We can apply the induction hypothesis to G_1. Now the vertices of G_1 are coloured blue and green and some of its edges are marked. If there is an edge in G connecting v with some vertex which is now green, we make v blue and mark this edge. If all the edges going from v lead to blue vertices, we make v green and mark any one of these edges. In both cases the marked edges form a connected graph with all the vertices of G, and obviously we have neither new marked edges connecting vertices of the same colour, nor edges of G connecting two green vertices.

Junior League

5. The answer is 10^6.

First we prove that the sum can not be greater than 10^6. All the numbers visible in the beginning do not exceed 10. After the first round of jumping the sum increased, that is, Kostya saw some new numbers. However, all these numbers did not exceed 100. Next time the sum increased again, that is, Kostya saw some even newer numbers which he could not see on previous two occasions. These newly-found numbers can not exceed 1000. Proceeding in the same way we see that after each round of jumping Kostya sees some numbers he never saw before. Since the frogs cover only 5 squares in the beginning, the number of jumps allowing the sum to increase so dramatically can not be more than 5.

The example can be given as follows. Let the frogs occupy the leftmost five squares in the last row, and every time each frog jumps right to the next square. Suppose the numbers under the frogs are a, b, c, d, f, and the number g is written in all the other squares. Putting $g = 10/95$, $a = 100 - g$, $b = 10^3 - a - 2g$, $c = 10^3 - b - a - 3g$, $d = 10^4 - a - b - c - 4g$, $f = 10^5 - a - b - c - d - 5g$, we obtain the desired result.

6. The answer is yes.

Lemma. Let Q, M be positive integers, $(M, 10) = 1$. Then the decimal representation of Q can be repeated several times so that the obtained number is divisible by M.

Proof. Among the numbers Q, QQ, QQQ, … there are two numbers leaving the same remainder when divided by M. We can take their difference and remove zeroes in the end. □

It suffices to produce a number N written by equal number of digits 1, 3, 5, 7, 9 and divisible by each 20-digit number written by odd digits.

Among all 20-digit numbers with odd digits we choose the number q divisible by the maximum degree of 5; let this degree be 5^ℓ. We can write q several times to get more than ℓ digits and then append digits to the left so that the resulting number Q has equal number of digits 1, 3, 5, 7, 9. Obviously Q is divisible by 5^ℓ.

Now we list all the 20-digit numbers with odd digits, divide each by the largest possible degree of 5, and multiply the results; let M be the product.

It follows from the lemma that repeating the decimal representation of Q we eventually get a number N divisible by M. This number satisfies all the conditions.

7. See problem 6, senior league.

8. If the groups are chosen, there exists r flights serving $2r$ cities and connecting cities from different groups. Therefore every choice of the group can be defined by selecting r flights serving $2r$ cities and choosing one end of each flight that belongs to the first group. The first selection can be made at most in $\binom{m}{r}$ ways, and the second one in 2^r ways. Thus the total number of choices does not exceed

$$\binom{m}{r}\cdot 2^r = \frac{m(m-1)\dots(m-r+1)}{r!}\cdot 2^r \leqslant m^r\cdot\frac{2^r}{r!} = m^r\cdot\frac{2}{1}\cdot\frac{2}{2}\cdot\ldots\cdot\frac{2}{r} \leqslant 2m^r.$$