Министерство образования Республики Саха (Якутия)

ЗАДАЧИ

IX МЕЖДУНАРОДНОЙ ОЛИМПИАДЫ ШКОЛЬНИКОВ ПО МАТЕМАТИКЕ, ИНФОРМАТИКЕ, ФИЗИКЕ И ХИМИИ

"Туймаада"

Якутск - 2002г.

МАТЕМАТИКА

MATHEMATICS

Senior league

1. Each of the points G and H lying from different sides of the plane of hexagon $ABCDEF$ is connected with all vertices of the hexagon. Is it possible to mark 18 segments thus formed by the numbers 1, 2, 3, …, 18 and arrange some real numbers at points A, B, C, D, E, F, G, H so that each segment is marked with the difference of the numbers at its ends?

2. The product of positive numbers a, b, c, and d is 1. Prove that

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+cd}{1+c}+\frac{1+da}{1+d}\geq 4.$$

3. A circle having common centre with the circumcircle of triangle ABC meets the sides of the triangle at six points forming convex hexagon $A_1A_2B_1B_2C_1C_2$ (A_1 and A_2 lie on BC, B_1 and B_2 lie on AC, C_1 and C_2 lie on AB). If A_1B_1 is parallel to the bisector of angle B, prove that A_2C_2 is parallel to the bisector of angle C.

4. A rectangular table with 2001 rows and 2002 columns is partitioned into 1×2 rectangles. It is known that any other partition of the table into 1×2 rectangles contains a rectangle belonging to the original partition. Prove that the original partition contains two successive columns covered by 2001 horizontal rectangles.

5. A positive integer c is given. The sequence $\{p_k\}$ is constructed by the following rule: p_1 is arbitrary prime and for $k \geq 1$ the number p_{k+1} is any prime divisor of $p_k + c$ not present among the numbers $p_1, p_2, \ldots, p_k$. Prove that the sequence $\{p_k\}$ cannot be infinite.

6. Find all the functions $f(x)$, continuous on the whole real axis, such that for every real x

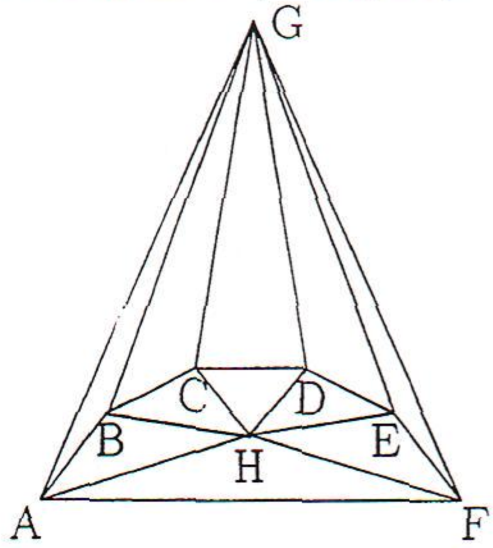
$$f(3x-2) \leq f(x) \leq f(2x-1).$$

7. The points D and E on the circumcircle of an acute triangle ABC are such that $AD = AE$. Let H be the common point of the altitudes of triangle ABC. It is known that $AH^2 = BH^2 + CH^2$. Prove that H lies on the segment DE.

8. A real number α is given. The sequence $n_1 < n_2 < n_3 < \ldots$ consists of all the positive integral n such that $\{n\alpha\} < \frac{1}{10}$. Prove that there are at most three different numbers among the numbers $n_2 - n_1, n_3 - n_2, n_4 - n_3, \ldots$

Junior league

1. Each of the points G and H is connected by non-intersecting segments with all vertices of hexagon $ABCDEF$ (see figure). Is it possible to mark 18 segments thus formed by the numbers 1, 2, 3, …, 18 and arrange some numbers (not necessarily integral) at points A, B, C, D, E, F, G, H so that each segment is marked with the difference of the numbers at its ends?



2. Points A_1, B_1, C_1 on the sides BC, CA, AB of triangle ABC respectively are such that

$$AC_1 : C_1B = BA_1 : A_1C = CB_1 : B_1A = 2 : 1.$$

Prove that if $A_1B_1C_1$ is equilateral then ABC is also equilateral.

3. Does there exist a quadratic trinomial such that all its values at positive integral points are integral degrees of 2?

4. A rectangular table with 2001 rows and 2002 columns is partitioned into 1×2 rectangles so that some two successive columns are covered by 2001 horizontal rectangles. Prove that *any other partition of the table into 1×2 rectangles contains a* rectangle belonging to the original partition.

5. For every $x, y \in [0; 1]$ prove the inequality

$$5(x^2 + y^2)^2 \leq 4 + (x + y)^4.$$

6. Pairwise different numbers are arranged in the squares of 100×100 table. Every minute each number is replaced by the largest of numbers in the neighbouring squares (i.e. having a common side with the square containing this number). Can the numbers become equal after 4 hours?

7. A positive integer c is given. The sequence $\{p_k\}$ is constructed by the following rule: p_1 is arbitrary prime and for $k \geq 1$ the number p_{k+1} is any prime divisor of $p_k + c$ not present among the numbers $p_1, p_2, \ldots, p_k$. Prove that the sequence $\{p_k\}$ cannot be infinite.

8. A circle with centre O touches the sides of an angle with vertex A at points K and M. A tangent to the circle meets the segments AK and AM at points B and C respectively; the line KM meets the segments OB and OC at points D and E. Prove that the area of triangle ODE is four times less than the area of triangle BOC if and only if $\angle A = 60°$.