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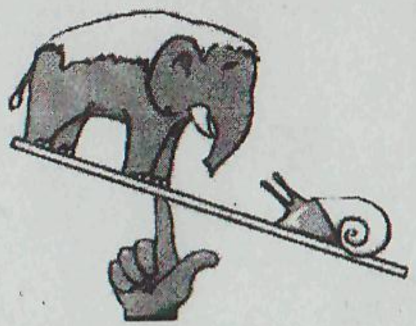
XXIII Международная олимпиада

«Туймаада»

Физика

Теоретический тур

Методическое пособие



Якутск, 15-22 июля 2016 г.

так что затратами теплоты на нагревание газа в какой-то момент можно будет вообще пренебречь по сравнению с затратами теплоты на испарение жидкости.

Примерная система оценивания

Применение закона теплопроводности	1
Вычисление T_3	1
Численный ответ для φ	1
Выражение для ΔQ_1	1
Вычисление P и P_a	1
Выражение для $\Delta Q_2'$	1
Выражение для Δm	1
Вычисление a	1
Выражение для $\Delta Q_2''$	1
Численный ответ для x	1

Junior league

Problem 1. Balls in infinite ocean

Suppose that the whole Universe is filled with an incompressible liquid with a density ρ_0, in which there are located at a large distance L from each other two uniform solid balls each having a volume V and densities ρ_1 and ρ_2.

1. With what force F_1 do the balls interact with each other?

2. Find vector sums F_{21} and F_{22} of fluid-pressure-induced forces on the first and second balls, respectively.

3. What external forces F_{31} and F_{32} must be applied to the first and second balls, respectively, to keep them at rest?

4. What is an acceleration a_1 of the first ball immediately after a simultaneous release of both balls, if an acceleration of the second ball is equal to a_2?

Problem 2. Gluck's solutions

Gluck, an experimenter, carried out experiments with dilute solutions: he took a U-shaped tube, divided it into two parts by a partition, which is permeable only to water, and poured pure water in one knee of tube, and in the other knee he poured a water solution of substance X (fig. 8). Gluck measured a turned out to be very small fraction x of substance X of total amount of matter in the right knee and found that surfaces of liquids in the knees of tube are located on different levels, moreover, a difference of heights Δh depends on a temperature T according to a formula $\Delta h = xRT/(\mu(g)$, where R is the universal gas constant, μ is the molar mass of water, g is the free fall acceleration.

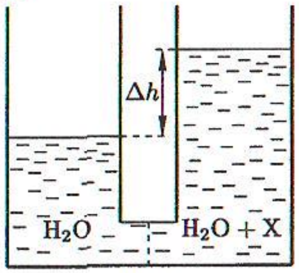


Fig. 8

1. Find a difference ΔP of pressures of liquids on different sides from the partition. The density ρ_0 of pure water and a maintained temperature T are known.

2. By what value ΔP_n is a saturated water vapor pressure above the solution less than a saturated vapor pressure P_n above the pure water? Suppose that the substance X does not evaporate.

3. By what value ΔT_k is a boiling temperature of solution at an atmospheric pressure P_0 greater than a boiling temperature of pure water? The boiling temperature T_k of pure water depends on a height z over sea level (in a range of small heights) according to a formula $T_k(z) = T_k(0) - \lambda z$, where λ is a known constant. A density of air ρ is known.

Problem 3. Bead on helix

A small bead was put on a long wire bent in a form of helix with a lead H, inscribed in a vertical cylinder of radius R, and the bead was released without an initial speed. A coefficient of friction between the wire and the bead is equal to μ. Find a steady speed v of bead.

Problem 4. Alternate grounding

Plates of spherical capacitor are two concentric metal spheres of radii r and R ($r < R$). The capacitor was charged to a voltage U_0, and then by means of flip-flop switch one started to alternately ground the capacitor plates (fig. 9). The inner plate was grounded as the first one, then the outer plate was grounded, then again the inner plate was grounded, and so on. As a result, each of plates was grounded n times. Find final charges q_n and Q_n on the inner and outer plates, respectively.

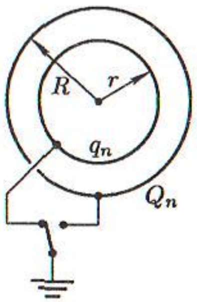


Fig. 9

Problem 5. Electromagnetic foil

Two fixed strips of well conductive foil are connected from one end to a source with an internal resistance r, and from the other end to universal terminals (fig. 10). If between the terminals nothing is connected, then the strips interact with each other with a force F_1, and if the terminals are short-circuited, then the strips interact with each other with a force F_2.

1. With what force F_3 will the strips interact with each other if a resistor of resistance R is connected to the terminals?
2. At what resistance R will this force be equal to zero?

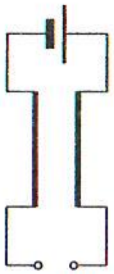


Fig. 10

Senior league

Problem 1. Undecided small ball

Two large smooth plates move codirectionally and perpendicular to their planes with constant velocities $\vec{u}_1$ and $\vec{u}_2$ (fig. 11). A small light ball moves between the plates elastically reflecting from them. At an initial moment of time a distance between the plates is equal to L_0, and the ball touches the overtaking plate and has a velocity $\vec{u}_0$ directed at an angle φ to the velocity $\vec{u}_1$ of the plate. It is known a relation between speeds: $u_0 \cos\varphi > u_1 > u_2 > 0$.

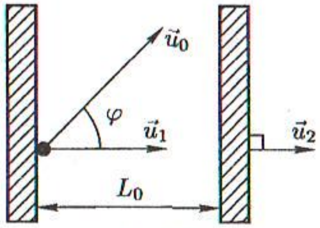


Fig. 11

1. Find a magnitude of displacement S_∞ of ball to a moment, when it makes very many collisions with the plates.
2. Find a magnitude of displacement S_n of ball to a moment immediately after an nth collision with the plate for any given n.

Problem 2. Particle in field

As a result of interaction only with a homogeneous magnetic field with a magnetic induction B a particle of mass m with a charge q flies with a constant in magnitude velocity v along a trajectory having a radius of curvature R.

1. Specify a condition, under which the given data are consistent.
2. Estimate a distance l, to which the particle will shift during a large time t.
3. Find a criterion of applicability of used approximation.
4. Determine the exact distance L, to which the particle will shift during an arbitrary time t.

Problem 3. Rearrangement of words

At what positive angle φ to the horizon is it necessary to throw a body so that during a time of its flight before its return to an original height an average of absolute value of vector of tangential acceleration coincides with an absolute value of average of vector of normal acceleration? Do not take into account air resistance.

Note. In this problem it is not required to give an answer in a general form, but it is necessary and sufficient to obtain the desired value with accuracy to tenths of angular degree.

Problem 4. Triangular plate

A plate in a form of regular triangle with a side $\ell\sqrt{2}$ is uniformly charged with a surface density σ. Find an electric field E at a point O located at the same distance ℓ from each vertex of plate.

Problem 5. Room thermodynamics

Problem A. In a window of a closed room there is mounted armoured glass of thickness $h_1 = 8$ cm, having a thermal conductivity coefficient $\varkappa_1 = 0.3$ W/(m·K). A temperature outdoors is $T_1 = -20$°C, and in the room a temperature $T_2 = 20$°C is maintained. Assume that near the window it is formed a transition layer of air of thickness $h_2 = 2$ cm, a temperature of which changes from the room temperature to a temperature of internal side of window. What relative humidity φ will be established in the room, if initially the air was humid enough? Take a thermal conductivity coefficient of air equal to $\varkappa_2 = 0.025$ W/(m·K).

Problem B. Find a ratio x of rates of drop of temperature from an initial value of $T = 15$°C immediately after turning off heating in a closed room with dry air and in another identical room with maximally humid air, if heat capacity of walls and items in the room is not taken into account. An atmospheric pressure $P_0 = 100$ kPa. A specific heat of evaporation of water at low temperatures is $L = 2.5$ MJ/kg.

Table 2

T, °C	P, Pa	T, °C	P, Pa
0	611.0	0	611.0
1	657.3	−1	568.6
2	705.8	−2	528.3
3	758.4	−3	490.6
4	813.4	−4	455.2
5	872.5	−5	422.2
6	935.1	−6	391.3
7	1002	−7	362.4
8	1073	−8	335.4
9	1148	−9	310.3
10	1228	−10	286.8
11	1313	−11	264.9
12	1402	−12	244.6
13	1498	−13	225.6
14	1598	−14	208.0
15	1706	−15	191.6
16	1818	−16	176.4
17	1938	−17	162.2
18	2064	−18	149.1
19	2198	−19	137.0
20	2338	−20	125.7

Note. In problems A and B the matter concerns different rooms. Values of saturated vapour pressure at some temperatures are given in table 2.

Possible solutions

Junior league

Problem 1. Balls in infinite ocean

From a definition of density we will express masses of balls:

$$m_1 = \rho_1 V, \qquad m_2 = \rho_2 V.$$

1. The balls interact between themselves only gravitationally according to the law of universal gravitation:

$$F_1 = \frac{\gamma m_1 m_2}{L^2} = \frac{\gamma \rho_1 \rho_2 V^2}{L^2}.$$

2. Mentally we will divide the entire liquid into a liquid ball of mass $m_0 = \rho_0 V$ located symmetrically to the second ball with respect to the center of the first one, and the rest of liquid. The total force of gravitational interaction of first ball and the rest of liquid is equal to zero, as the rest of liquid is symmetric with respect to the center of the first ball, i.e., for every particle of the rest of liquid another particle will be found, which acts on the first ball with the opposite gravitational force. We will assume that the direction from the center of the first ball to the center of the second ball is positive, then the free fall acceleration produced in the region of the first ball by the gravitational field of the second and liquid balls has the form

$$g_1 = \frac{\gamma m_2}{L^2} - \frac{\gamma m_0}{L^2} = \frac{\gamma(\rho_2 - \rho_0)V}{L^2}.$$

By virtue of condition $L^3 \gg V$ this gravitational field can be considered homogeneous, i.e., the same at any point near the first ball.

The vector sum of forces of pressure of liquid (or gas) on a body at rest in a medium — this is one of definitions of the Archimedes buoyant force, which can baffle except for those who do not like to remember verbal definitions of physical concepts, therefore, the required force F_{21} is found by means of weight of displaced fluid:

$$F_{21} = -m_0 g_1 = \frac{\gamma \rho_0 (\rho_0 - \rho_2) V^2}{L^2}.$$

Repeating the same arguments for the second ball (or just replacing indices), we obtain the free fall acceleration in the region of the second ball

$$g_2 = \frac{\gamma m_1}{L^2} - \frac{\gamma m_0}{L^2} = \frac{\gamma(\rho_1 - \rho_0)V}{L^2}$$

and the answer to the second question of this point:

$$F_{22} = -m_0 g_2 = \frac{\gamma \rho_0 (\rho_0 - \rho_1) V^2}{L^2}.$$

3. To keep the first ball at rest, it is required to apply to it a force, which is equal in magnitude and directed oppositely to the sum of forces acting on it from the sides of liquid and second ball:

$$F_{31} = -(m_1 g_1 + F_{21}) = -\frac{\gamma(\rho_1 - \rho_0)(\rho_2 - \rho_0)V^2}{L^2}.$$

Repeating the same arguments for the second ball (or just replacing the indices), we get the answer to the second question of this point:

$$F_{32} = -(m_2 g_2 + F_{22}) = -\frac{\gamma(\rho_1 - \rho_0)(\rho_2 - \rho_0)V^2}{L^2}.$$

Note. We will note that the found forces were the same (only in the absolute value since the positive directions for them were chosen in the opposite directions). This coincidence could be foreseen beforehand, mentally connecting the balls by a thin light rod: it would act on them with forces of the same magnitude, and the balls would remain at rest.

Note. It deserves special attention a method of finding the forces F_{31} and F_{32} by subtracting the density ρ_0 from the whole space, after which a void remains in the place of liquid, and the densities of balls become $(\rho_1 - \rho_0)$ and $(\rho_2 - \rho_0)$, respectively. The strength of gravitational interaction of new balls with the required sign will be the answer to both questions of this point. This force is also called an effective force of interaction, and it can be found in practice, for example, in a calculation of interaction of electrons in a crystal lattice: they do not just repel each other according to Coulomb's law, but each of them causes displacements of other charges, indirectly changing the resulting force acting on the second electron, therefore, the total change of resulting force acting on one particle induced by emergence of second particle is called the effective strength of interaction. The effective strength of interaction (taking into account influence of environment) is not to be confused with the usual force of interaction, which we found in this problem in the first point.

4. Why in the problem statement is the acceleration of second ball given, if we know the masses of balls and all forces anyway? The fact is in existence of a so-called added-liquid mass, i.e., the mass of liquid, which comes into motion with the body, since the medium needs to make way for the body and to fill space behind it. The added-liquid mass depends only on a shape and sizes of moving body and the density of liquid, therefore it will be the same for both balls. Taking into account the added-liquid mass μ and the resulting force found in the previous point

$$F_3 = \frac{\gamma(\rho_1 - \rho_0)(\rho_2 - \rho_0)V^2}{L^2}$$

laws of motion of balls immediately after their release will take a form

$$F_3 = (m_1 + \mu)a_1, \qquad F_3 = (m_2 + \mu)a_2,$$

from which after exclusion of μ we find the acceleration of the first ball:

$$a_1 = \frac{F_3 a_2}{F_3 + (m_1 - m_2)a_2}.$$

Note. We will verify the obtained formula in particular cases. If the balls are the same ($m_1 = m_2$), then the formula gives a quite expected equality $a_1 = a_2$. If the liquid was not present, then for redundant data a condition $F_3 = m_2 a_2$ was supposed to be satisfied, after a substitution of which the formula is simplified to an also quite expected form $a_1 = F_3/m_1$.

Grading system

The answer for F_1 1
The answers for F_{21} and F_{22} 3
The answers for F_{31} and F_{32} 3
The answer for a_1 3

Problem 2. Gluck's solutions

1. A density of dilute solution ($x \ll 1$) can be considered as equal to the density of pure water, therefore, the desired difference of pressures has a form

$$\Delta P \approx \rho_0 g \Delta h = \frac{\rho_0 x RT}{\mu}.$$

2. From the Mendeleev-Clapeyron equation (the ideal gas law) we will express the density of saturated vapor above the pure water:

$$\rho_n = \frac{\mu P_n}{RT}.$$

The decrease of saturated vapor pressure above the solution can be calculated according to a formula of hydrostatic pressure of column of vapor:

$$\Delta P_n = \rho_n g \Delta h = \frac{\mu P_n}{RT} \cdot g \cdot \frac{xRT}{\mu g} = P_n x. \qquad (28)$$

3. A boiling temperature is a temperature, at which a saturated vapor pressure is equal to an external pressure. The saturated vapor pressure above the solution is less than the saturated vapor pressure above the pure water, but it grows with an increase of temperature. Therefore, it is necessary to find such the increase ΔT_k to the temperature, which compensates for an effect of consideration of solution (but not the pure water).

An increase of pressure of saturated vapors at the height $z < 0$ (below sea level) is equal to an additional pressure of column of air of height $(-z)$:

$$\Delta P_n = -\rho g z = \rho g \frac{\Delta T_k}{\lambda}, \qquad (29)$$

where the quantity z is expressed from the dependence $T_k(z)$ given in the statement of problem. Equating formulas (28) and (29) for ΔP_n and taking into account that at the boiling temperature the equality $P_n = P_0$ is satisfied, we find the required temperature difference:

$$\Delta T_k = \frac{\lambda P_0}{\rho g} x.$$

Grading system

The answer for ΔP 2
The density of saturated vapor 1
The answer for ΔP_n 2
The equality $P_n = P_0$ at the boiling temperature 1
The idea of equality ΔP_n from the two effects 1
The expression ΔP_n through ΔT_k 1
The answer for ΔT_k 2

Problem 3. Bead on helix

We will consider a vertical plane α that touches a cylinder, which is circumscribed around the helix, at a point, in which the bead is located at some instant of time (fig. 12). The bead of mass m moving along the wire with the constant speed v is acted upon by three forces, which are constant in magnitude: a force of gravity mg and a friction force F, which lie in the plane α, and a normal reaction force N, which does not lie in the plane α, and therefore in fig. 12 it itself is not depicted, but its projection $N_\parallel$ on the plane α is pictured.

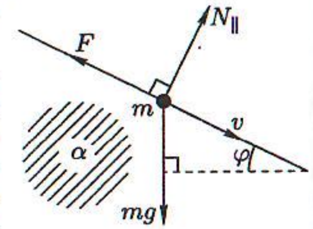


Fig. 12

Since the bead has only centripetal acceleration, which is perpendicular to the plane α and is created by a perpendicular to the plane α component $N_\perp$ of normal reaction, one can write in the plane α conditions of equality of forces on a direction of velocity and perpendicular to it:

$$F = mg\sin\varphi, \qquad N_\parallel = mg\cos\varphi, \tag{30}$$

where the angle φ of slope of trajectory to the horizon can be found from geometric sizes of helix:

$$\tan\varphi = \frac{H}{2\pi R}. \tag{31}$$

In a system of reference moving downward with a speed $v\sin\varphi$, the bead moves with a speed $v\cos\varphi$ around a circumference of radius R, which allows finding a force that produces the centripetal acceleration:

$$N_\perp = m\cdot\frac{(v\cos\varphi)^2}{R}. \tag{32}$$

After writing a formula for the sliding friction force

$$F = \mu N \tag{33}$$

and the Pythagorean theorem for the normal reaction

$$N = \sqrt{N_\parallel^2 + N_\perp^2} \tag{34}$$

a physical part of problem is finished, and it remains to combine all the equations in a system and to solve it for v.

We will square equation (33) and substitute expressions (34), (30) and (32):

$$(mg\sin\varphi)^2 = \mu^2\left((mg\cos\varphi)^2 + \left(\frac{mv^2\cos^2\varphi}{R}\right)^2\right),$$

cancel $m^2g^2\cos^2\varphi$ for simplification:

$$\tan^2\varphi = \mu^2\left(1 + \frac{v^4\cos^2\varphi}{g^2R^2}\right),$$

use a trigonometric identity $1/\cos^2\varphi = 1 + \tan^2\varphi$:

$$v^4 = g^2R^2(1+\tan^2\varphi)\left(\frac{\tan^2\varphi}{\mu^2} - 1\right),$$

substitute formula (31) and obtain a final answer:

$$v = \sqrt{gR}\cdot\sqrt[4]{\left(1 + \frac{H^2}{4\pi^2R^2}\right)\left(\frac{H^2}{4\pi^2R^2\mu^2} - 1\right)}.$$

We will note that under a condition $H < 2\pi R\mu$ the radicand is negative, which is nonphysical and evidences a violation of assumption (33) that the bead will be acted upon by exactly the sliding friction force, but not a static friction force. So, it is necessary to add to the answer one more case: $v = 0$ for $H < 2\pi R\mu$.

Grading system

The formula for $\tan\varphi$ 1
The expression for F 1
The expression for $N_\parallel$ 1
The expression for $N_\perp$ 2
The connection between F and N 1
The Pythagorean theorem for N 1
The final answer 2
The condition when the bead is at rest 1

Problem 4. Alternate grounding

Let q_0 and Q_0 be initial charges on the inner and outer plates, respectively, then from a condition of electroneutrality of capacitor $q_0 = -Q_0$. We will express the voltage between the plates as a difference of their potentials:

$$U_0 = \varphi_0^{outer} - \varphi_0^{inner} = \left(\frac{kQ_0}{R} + \frac{kq_0}{R}\right) - \left(\frac{kQ_0}{R} + \frac{kq_0}{r}\right) =$$

$$= kq_0\left(\frac{1}{R} - \frac{1}{r}\right) = kQ_0\left(\frac{1}{r} - \frac{1}{R}\right),$$

from which

$$Q_0 = \frac{Rr}{k(R-r)} \cdot U_0. \qquad (35)$$

Note. The obtained expression could be written at once using a formula for capacitance of spherical capacitor:

$$C = \frac{Rr}{k(R-r)}.$$

When a conductor is grounded, its potential becomes equal to a potential of infinitely remote body, which in accordance with a formula $\varphi(x) = kq/x$ is equal to zero. Therefore, the charge of plate after each ground can be found from the condition of equality to zero of its potential:

$$\varphi_1^{inner} = \frac{kQ_0}{R} + \frac{kq_1}{r} = 0, \qquad \text{from which} \qquad q_1 = -\frac{r}{R}Q_0;$$

$$\varphi_1^{outer} = \frac{kQ_1}{R} + \frac{kq_1}{R} = 0, \qquad \text{from which} \qquad Q_1 = -q_1 = \frac{r}{R}Q_0.$$

A calculation of charges after the second pair of grounds is completely analogous:

$$\varphi_2^{inner} = \frac{kQ_1}{R} + \frac{kq_2}{r} = 0, \qquad \text{from which} \qquad q_2 = -\frac{r}{R}Q_1 = -\left(\frac{r}{R}\right)^2 Q_0;$$

$$\varphi_2^{outer} = \frac{kQ_2}{R} + \frac{kq_2}{R} = 0, \qquad \text{from which} \qquad Q_2 = -q_2 = \left(\frac{r}{R}\right)^2 Q_0.$$

Generalizing the results for n pairs of grounds and substituting the expression for Q_0 from formula (35), we will finally obtain

$$q_n = -\left(\frac{r}{R}\right)^n \cdot \frac{RrU_0}{k(R-r)}, \qquad Q_n = \left(\frac{r}{R}\right)^n \cdot \frac{RrU_0}{k(R-r)}.$$

We will note that it is not said in the problem statement, which of plates was initially positively charged, and which was negatively charged. In the solution we assumed that $Q_0 > 0$, but $q_0 < 0$. But if this is not so, all the answers will change the signs.

In order to answer the second question of problem, we will equate the expression for F_3 to zero, whence we will find the required resistance, at which the electric and magnetic interactions compensate each other:

$$R = r\sqrt{\frac{F_2}{F_1}}.$$

Grading system

The obtainment of proportionality $F_1 = aU^2$	2
The obtainment of proportionality $F_2 = aI^2$	2
The use of Ohm's law	1
Taking into account the direction of forces	1
The answer for F_3	2
The answer for R	2

Senior league

Problem 1. Undecided small ball

1. Since at the collisions of ball with the plates there is no friction, a component of velocity of ball, which perpendicular to the direction of movement of plates, will remain constant:

$$v_\perp = u_0 \sin\varphi.$$

As the plates approach each other, the collisions of ball will happen more often and their number will limitlessly increase, therefore, the value S_∞ can be found by considering a situation when the distance between the plates will be negligible, i.e., the plates will almost collide from which we find a shift x_∞ of overtaking plate to the time t_∞ of their encounter:

$$t_\infty = \frac{L_0}{u_1 - u_2}, \qquad x_\infty = u_1 t_\infty = \frac{L_0 u_1}{u_1 - u_2}.$$

A shift of ball in the transverse direction has a form

$$y_\infty = v_\perp t_\infty = \frac{L_0 u_0 \sin\varphi}{u_1 - u_2}.$$

An answer to the first question we get according to the Pythagorean theorem:

$$S_\infty = \sqrt{x_\infty^2 + y_\infty^2} = \frac{L_0}{u_1 - u_2}\sqrt{u_1^2 + u_0^2 \sin^2\varphi}.$$

2. To simplify calculations, we will go to a system of reference, moving in the direction of movement of plates with a speed $v_c = (u_1 + u_2)/2$ and meanwhile we will only consider a longitudinal motion of ball. In the new system of reference the plates come closer with equal in magnitude speeds $u = (u_1 - u_2)/2$, and the ball has an initial velocity $v_0 = u_0 \cos\varphi - v_c$.

Let v_n be a speed of ball immediately after the nth collision, then from a condition of elasticity of collision (the speed of ball relative to the plate does not change as a result of collision) one can write a recurrence relation

$$v_n = v_{n-1} + 2u,$$

from which a general formula follows

$$v_n = v_0 + 2un.$$

Let L_n be a distance between the plates immediately after the nth collision, τ_n is a time of motion of ball with the speed v_n, i.e., the time between collisions with numbers n and $n+1$, then from formulas of uniform motion one can write the following equations:

$$\tau_n = \frac{L_n}{v_n + u} = \frac{L_n}{v_0 + 2un + u}, \tag{40}$$

$$L_n = L_{n-1} - 2u\tau_{n-1} = L_{n-1} - \frac{2uL_{n-1}}{v_0+2un-u} = L_{n-1}\cdot\frac{v_0+2un-3u}{v_0+2un-u}.$$

We will use the last formula, successively expressing L_n through L_{n-1}, then L_{n-1} through L_{n-2}, and so on to $L_1 = L_0(v_0-u)/(v_0+u)$:

$$L_n = \frac{v_0+2un-3u}{v_0+2un-u}\cdot\frac{v_0+2un-5u}{v_0+2un-3u}\cdot\ldots\cdot\frac{v_0+u}{v_0+3u}\cdot\frac{v_0-u}{v_0+u}\cdot L_0,$$

from which after cancellation we obtain a general formula

$$L_n = \frac{(v_0-u)L_0}{v_0+2un-u}. \qquad (41)$$

Note. It is always useful to check nontrivial general formulas (even intermediate ones) for special cases. When $n=0$, formula (41) gives a correct identity $L_0 = L_0$. When $v_0 = u$, formula (41) gives $L_1 = 0$, which is completely expected, as in this case the ball moves with the overtaking plate and its first collision with the other plate will simultaneously be the collision of plates. When $v_0 \gg 2un$, formula (41) gives $L_n \approx L_0$, which is fully expected, as in this case the plates will almost not move during a time of all n collisions of ball.

Substituting expression (41) in equality (40), we obtain a general formula

$$\tau_n = \frac{(v_0-u)L_0}{(v_0+2un-u)(v_0+2un+u)},$$

which we will rewrite in a more convenient for summing form:

$$\tau_n = \frac{(v_0-u)L_0}{2u}\left(\frac{1}{v_0+2un-u} - \frac{1}{v_0+2un+u}\right),$$

in order to find a moment of time of nth collision:

$$t_n = \tau_0+\tau_1+\ldots+\tau_{n-1} = \frac{(v_0-u)L_0}{2u}\left(\left(\frac{1}{v_0-u}-\frac{1}{v_0+u}\right)+\right.$$

$$\left.+\left(\frac{1}{v_0+u}-\frac{1}{v_0+3u}\right)+\ldots+\left(\frac{1}{v_0+2un-3u}-\frac{1}{v_0+2un-u}\right)\right)=$$

$$=\frac{(v_0-u)L_0}{2u}\left(\frac{1}{v_0-u}-\frac{1}{v_0+2un-u}\right) = \frac{nL_0}{v_0+2un-u}. \qquad (42)$$

Note. In the usual way we check the nontrivial general formula for special cases. When $n=0$, formula (42) gives $t_0 = 0$, which is completely expected, as the initial touch of ball and plate can be considered as a zero collision. When $n\to\infty$, formula (42) gives $t_\infty = L_0/(2u)$, which coincides with the result from the first point of problem. When $u=0$, formula (42) gives $t_n = nL_0/v_0$, which is entirely

expected, since with fixed plates the ball goes through between the collisions the same path L_0.

We will return to an original system of reference. Immediately after an even collision the ball is near the overtaking plate, which has at this moment a shift u_1t_n, and immediately after an odd collision the ball is near the other plate, which is away from the overtaking one by L_n. With the use of modulo operation the shift of ball along the direction of movement of plates is written as one formula:

$$x_n = u_1t_n + L_n\cdot(n \bmod 2) = \frac{u_1nL_0}{v_0+2un-u} + \frac{(v_0-u)L_0}{v_0+2un-u}\cdot(n \bmod 2) =$$

$$= L_0\cdot\frac{u_1n + (u_0\cos\varphi - u_1)(n \bmod 2)}{u_0\cos\varphi - u_1 + (u_1-u_2)n}.$$

The shift of ball in the transverse direction has a form

$$y_n = v_\perp t_n = \frac{L_0nu_0\sin\varphi}{u_0\cos\varphi - u_1 + (u_1-u_2)n}.$$

An answer to the second question we get according to the Pythagorean theorem:

$$S_n = \sqrt{x_n^2+y_n^2} = \frac{L_0\sqrt{(u_1n+(u_0\cos\varphi-u_1)(n \bmod 2))^2 + n^2u_0^2\sin^2\varphi}}{u_0\cos\varphi - u_1 + (u_1-u_2)n}.$$

Grading system

The answer for S_∞ 2
The general formula for v_n 1
The general formula for L_n 2
The general formula for τ_n 2
The general formula for t_n 2
The answer for S_n 1

Problem 2. Particle in field

The charged particle moves in the homogeneous magnetic field in a helix as a result of addition of two movements: a motion along the field with a constant speed $v_\parallel$ and a motion in a plane perpendicular to the field in a circle of radius r with a constant speed $v_\perp$.

1. The particle moves under action of the Lorentz force $F = qv_\perp B$, therefore from Newton's second law $F = ma$ we will express a particle acceleration a:

$$a = \frac{qv_\perp B}{m}. \qquad (43)$$

Substituting the obtained expression into a formula for a centripetal acceleration $a = v^2/R$, we will obtain the component of particle velocity perpendicular to the field:

$$v_\perp = \frac{mv^2}{qBR}. \qquad (44)$$

By the Pythagorean theorem $v^2 = v_\parallel^2 + v_\perp^2$ we find the component of particle velocity along the field:

$$v_\parallel = v\sqrt{1 - \frac{m^2v^2}{q^2B^2R^2}}.$$

The given data are consistent, if the radicand is nonnegative, that is, under the condition $mv \leqslant qBR$.

2. The circular motion is limited, therefore its contribution to the total displacement after the large time can be neglected as compared to the displacement along the field, from which it follows the estimate for the desired shift:

$$l \approx v_\parallel t = vt\sqrt{1 - \frac{m^2v^2}{q^2B^2R^2}}. \tag{45}$$

3. In an inertial reference system moving with the speed $v_\parallel$ along the field, the particle moves round the circle of radius r with the constant speed $v_\perp$ and the same acceleration a, therefore from the formula of centripetal acceleration $a = v_\perp^2/r$ after substituting expressions (43) and (44) we find the circle radius:

$$r = \frac{v_\perp^2}{a} = \frac{mv_\perp}{qB} = \frac{m^2v^2}{q^2B^2R}. \tag{46}$$

The approximation used for finding l is correct when $l \gg r$, from which after substituting expressions (45) and (46) we find the criterion of applicability:

$$t \gg \frac{m^2v}{q^2B^2R}\left(1 - \frac{m^2v^2}{q^2B^2R^2}\right)^{-1/2}.$$

Note. A simplified criterion «the time is much greater than the period of rotation», leading to a relationship $qBt \gg 2\pi m$, is insufficient for an estimation of applicability of used approximation, as it does not take into account a ratio of velocities: the particle can rotate for a long time, but because of small component $v_\parallel$ the particle can almost not be shifted as a result.

4. For finding L it is necessary to *take into account the transverse* displacement of particle. During the time t it will be displaced along the circumference by an angle $\varphi = v_\perp t/r$. A chord subtending this arc has a length

$$l_\perp = 2r\left|\sin\frac{\varphi}{2}\right| = \frac{2m^2v^2}{q^2B^2R}\left|\sin\left(\frac{qB}{2m}t\right)\right|.$$

By the Pythagorean theorem we find the total displacement:

$$L = \sqrt{l^2 + l_\perp^2} = \sqrt{\left(1 - \frac{m^2v^2}{q^2B^2R^2}\right)v^2t^2 + \frac{4m^4v^4}{q^4B^4R^2}\sin^2\left(\frac{qB}{2m}t\right)}.$$

Grading system

Understanding of character of particle motion	1
The expression for a	1
The expression for $v_\perp$	1
The expression for $v_\parallel$	1
The condition of consistency of given data	1
The approximate expression for l	1
The expression for r	1
The criterion of applicability of approximation	1
The expression for $l_\perp$	1
The exact expression for L	1

Problem 3. Rearrangement of words

The absolute value of vector of tangential acceleration is responsible for a change of speed, therefore for finding its average value it is sufficient to divide the total change of speed in the course of rise of body to the highest point by a time of rise, so an average over a time of descent will be the same by virtue of symmetry of movement, however, it would be a mistake to apply this logic directly to the whole flight as the speed nonmonotonically depends on the time, changing a decrease to an increase at the highest point of trajectory.

Let v_0 be the initial velocity of body, g is the free fall acceleration, then the time of rise is calculated according a formula $T = v_0\sin\varphi/g$, and the average of absolute value of vector of tangential acceleration has a form

$$\langle|\vec{a}_t|\rangle = \frac{v_0 - v_0\cos\varphi}{T} = g\cdot\frac{1-\cos\varphi}{\sin\varphi}. \tag{47}$$

A calculation of absolute value of average of vector of normal acceleration is fundamentally different from the previous computations, and by not only that *another component of acceleration is considered*, but yet in that now at first the averaging is carried out, and then the absolute value is taken.

By virtue of symmetry of movement relative to a vertical straight line passing through the highest point of trajectory, the average of vector of normal acceleration will be directed vertically down, therefore, only its vertical projection a_{ny} can immediately be averaged, and only during the time of rise, as during the time of descent the average of projection will be the same. Let α be a slope angle of velocity to the horizon at a time t from the start of flight, then the desired average value has a form

$$|\langle\vec{a}_n\rangle| = |\langle a_{ny}\rangle| = \frac{1}{T}\int_0^T g\cos^2\alpha dt = \frac{1}{T}\int_0^T \frac{gdt}{\mathrm{tg}^2\alpha + 1} =$$

$$= \frac{1}{T}\int_0^T \frac{gdt}{\left(\frac{v_0\sin\varphi - gt}{v_0\cos\varphi}\right)^2 + 1} = \frac{v_0\cos\varphi}{T}\cdot\mathrm{arctg}\left(\frac{gt - v_0\sin\varphi}{v_0\cos\varphi}\right)\Big|_0^T =$$

$$= \frac{g\cos\varphi}{\sin\varphi}\cdot(\operatorname{arctg}0 - \operatorname{arctg}(-\operatorname{tg}\varphi)) = \frac{g\varphi\cos\varphi}{\sin\varphi}. \qquad (48)$$

Equating final expressions (47) and (48), we obtain after cancellation of $g\operatorname{ctg}\varphi$ an equation

$$\frac{1}{\cos\varphi} - 1 = \varphi. \qquad (49)$$

This is a transcendental equation, so we can solve it only by numerical methods. To determine a number of its solutions in an interval $0 < \varphi < \pi/2$, which is of interest to us, we will plot in fig. 13 graphs of left (a solid line) and right (a dotted line) sides of equation (49).

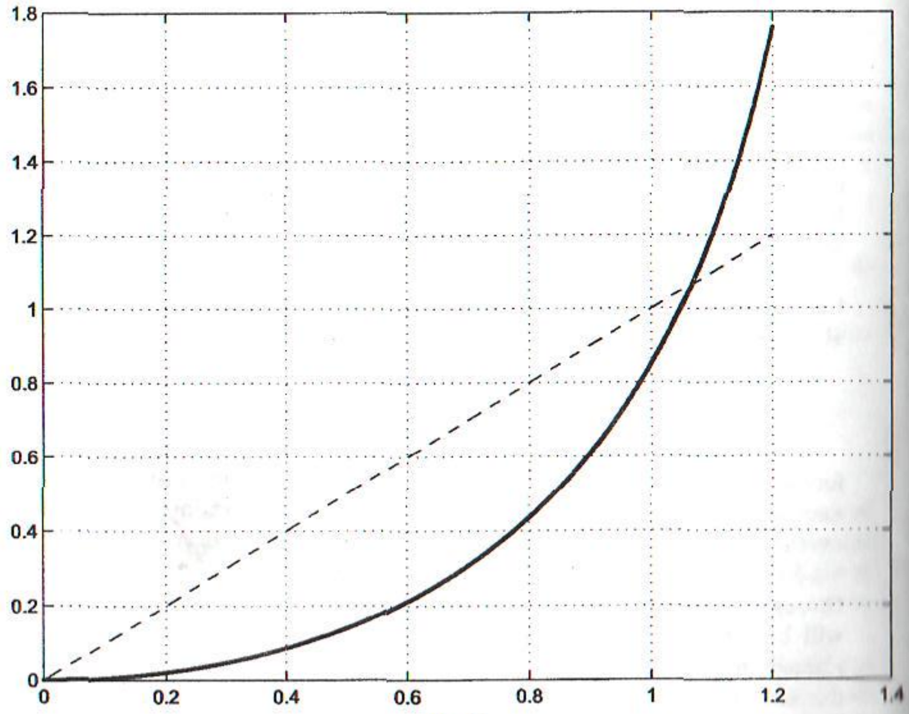


Fig. 13

From a draft of graph (participants of Olympiad may not use programs for plotting, therefore, we will not try to determine an answer immediately from the graph) it can be seen that the equation has in the interval $0 < \varphi < \pi/2$ the only root near a point $\varphi \approx 1$, that can be found without the graph, if one notices that $\cos 1 \approx \cos 60^\circ = 0.5$.

In order to improve the accuracy of obtained estimate $\varphi \approx 1$, we will use a method of successive approximations writing equation (49) in a form of function

$$\varphi_n = f(\varphi_{n-1}) = \frac{1}{\cos\varphi_{n-1}} - 1. \qquad (50)$$

Taking as a zero-order approximation $\varphi_0 = 1$, we will find the following:

$$\varphi_1 = f(\varphi_0) \approx 0.851,$$
$$\varphi_2 = f(\varphi_1) \approx 0.517,$$
$$\varphi_3 = f(\varphi_2) \approx 0.150,$$
$$\varphi_4 = f(\varphi_3) \approx 0.011.$$

It is seen that the sequence more and more moves from the assumed position of root, so, function (50) was poorly chosen, therefore, we will rewrite equation (49) in another way:

$$\varphi_n = F(\varphi_{n-1}) = \arccos\left(\frac{1}{\varphi_{n-1}+1}\right). \qquad (51)$$

Taking as the zero-order approximation $\varphi_0 = 1$, we will find the following:

$$\varphi_1 = F(\varphi_0) \approx 1.047,$$
$$\varphi_2 = F(\varphi_1) \approx 1.060,$$
$$\varphi_3 = F(\varphi_2) \approx 1.064,$$
$$\varphi_4 = F(\varphi_3) \approx 1.065,$$
$$\varphi_5 = F(\varphi_4) \approx 1.065.$$

Since the equality $\varphi_5 = \varphi_4$ is fulfilled with the sufficient accuracy, the final answer can be written:

$$\varphi \approx 1.065 \approx 61.0^\circ.$$

Note. We will note that the result of application of method of successive approximations significantly depends on the selected form of writing the equation in the form of function. If the sequence of approximate values tends to the desired value, then they say that the method converges, but otherwise — that the method diverges. It is described in a course of computational mathematics why it happens like this and how to select «correct» functions.

Grading system

The expression for $\langle|\vec{a}_t|\rangle$ 2
The expression for $|\langle\vec{a}_n\rangle|$ 5
The obtainment of zero-order approximation $\varphi_0 \approx 1$ 1
The obtainment of numerical answer $\varphi \approx 1.065 \approx 61.0^\circ$ 2

Problem 4. Triangular plate

General formula. At first we will derive a useful general formula

$$F_\perp = \sigma\Phi, \qquad (52)$$

where σ is the surface density of charge of uniformly charged plate, Φ is a flux of external electric field (not necessarily homogeneous) through the plate, $F_\perp$ is a perpendicular to the plate projection of force, acting on the plate from the external electric field.

On a small piece of plate having an area dS and a charge $dq = \sigma dS$ a force $dF_\perp = E_\perp dq$ acts perpendicular to the plate. An integration of $dF_\perp$ over the whole plate of area S with the use of definition of flux Φ gives the stated formula:

$$F_\perp = \int_S dF_\perp = \sigma \int_S E_\perp dS = \sigma\Phi.$$

Solution of problem. We will place a test charge q at the point O and write the desired electric field as $E = F/q$, where F is the force acting on the test charge from the plate. According to Newton's third law the same in magnitude force F acts on the plate from the test charge. By virtue of system symmetry of rotation by an angle $2\pi/3$ about an axis passing through the point O and the center of plate, the force F is directed along this axis, from which an equality $F = F_\perp$ follows. Taking into account formula (52), we obtain an expression

$$E = \frac{\sigma\Phi}{q}, \qquad (53)$$

where Φ is the flux of electric field of test charge through the plate.

We will note that the plate and the test charge can be placed inside and on a surface of the cube as shown in fig. 14. Since 8 touching cubes with the common vertex O completely surround the charge q, the flux Φ is 1/8 of total flux Φ_0 from the charge q, which according to Gauss's theorem has a form $\Phi_0 = q/\varepsilon_0$.

A substitution of expression $\Phi = q/(8\varepsilon_0)$ in formula (53) gives the final answer:

$$E = \frac{\sigma}{8\varepsilon_0}.$$

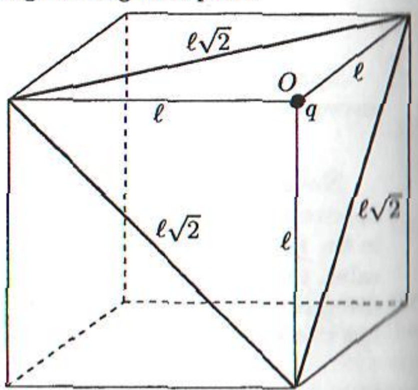


Fig. 14

Grading system

The use of formula $E = F/q$ 1
The use of formula $F = F_\perp$ 1
The use of formula $F_\perp = \sigma\Phi$ 2
The use of formula $\Phi = \Phi_0/8$ 2
The use of formula $\Phi_0 = q/\varepsilon_0$ 1
The final answer 3

Problem 5. Room thermodynamics

Задача *A*. Let T_3 be the temperature of internal side of window, S is an area of window, then an equality of heat fluxes through the glass and through the boundary layer of air has a form

$$\varkappa_1 \cdot S \cdot \frac{T_3 - T_1}{h_1} = \varkappa_2 \cdot S \cdot \frac{T_2 - T_3}{h_2},$$

from which we obtain an equality

$$T_3 = \frac{\frac{\varkappa_1}{h_1} \cdot T_1 + \frac{\varkappa_2}{h_2} \cdot T_2}{\frac{\varkappa_1}{h_1} + \frac{\varkappa_2}{h_2}} \approx -10^\circ\text{C}.$$

The vapour will condense on the glass until its pressure will become equal to the saturated vapour pressure at the temperature of glass. Hence we find the steady air humidity in the room:

$$\varphi = \frac{P_{sat}(T_3)}{P_{sat}(T_1)} = \frac{286.8\ \text{Pa}}{2338\ \text{Pa}} \approx 12.3\%.$$

Problem *B*. The required ratio we will find according to a formula $x = \Delta Q_2/\Delta Q_1$, where ΔQ_1 and ΔQ_2 are amounts of heat, which it is necessary to bring to the dry and maximally humid air, correspondingly, to heat by a small temperature ΔT. Let V be a volume of room, ν is an amount of air in the room, R is the universal gas constant, then with the use of the Mendeleev-Clapeyron equation (the ideal gas law) one can write the first law of thermodynamics in a form

$$\Delta Q_1 = \frac{5}{2}\nu R\Delta T = \frac{5}{2} \cdot \frac{P_0 V}{T} \cdot \Delta T. \qquad (54)$$

Since the air is maximally humid, we find the vapour pressure P in the table given in the problem statement, and a pressure P_a of air is expressed as a partial pressure:

$$P = P_{sat}(T) = 1706\ \text{Pa}, \qquad P_a = P_0 - P = 98.3\ \text{kPa}.$$

An expression for the humid air, which is analogous to formula (54), has a form

$$\Delta Q_2' = \frac{5}{2} \cdot \frac{P_a V}{T} \cdot \Delta T + 3 \cdot \frac{PV}{T} \cdot \Delta T,$$

where in writing numerical coefficients it is taken into account that the water vapour is a triatomic gas, but it is not yet taken into account evaporation of water. From the Mendeleev-Clapeyron equation *we will express a mass of vapour* and its change:

$$m = \frac{\mu P_{sat} V}{RT}, \qquad \Delta m = m'(T)\Delta T = \frac{\mu V}{RT^2}(T \cdot P'_{sat}(T) - P_{sat})\Delta T.$$

We will numerically find the derivative *from the previous formula* using the table:

$$a = P'_{sat}(T) = \frac{P_{sat}(T + 1\ \mathrm{K}) - P_{sat}(T - 1\ \mathrm{K})}{2\ \mathrm{K}} = 110\ \mathrm{Pa/K}.$$

Note. We will note that because of finiteness of Δx formulas of numerical differentiation

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \text{и} \qquad f'(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

give different results usually lying on opposite sides of the true value, therefore, a *more precise formula for numerical differentiation* is their arithmetic average:

$$f'(x) = \frac{1}{2}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{f(x) - f(x-\Delta x)}{\Delta x}\right) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x},$$

which we used in the calculation of a. It is described in a course of computational mathematics how to choose an optimal formula for *a specific function that* is especially important for higher-order derivatives.

An amount of heat, which is necessary for evaporation, has a form

$$\Delta Q''_2 = L\Delta m = \frac{\mu V}{RT^2}(aT - P)\Delta T.$$

Using an expression $\Delta Q_2 = \Delta Q'_2 + \Delta Q''_2$, we will obtain the required ratio:

$$x = \frac{P_a}{P_0} + \frac{6P}{5P_0} + \frac{2\mu L}{5RTP_0}(aT - P).$$

Each term in this formula is associated with a separate effect, therefore, for their quantitative comparison we will calculate x step-by-step (not forgetting during a substitution to convert the temperature to kelvins):

$$x = 0.983 + 0.0205 + 2.254 \approx 3.26.$$

Thus, the dry air cools much faster than the humid one.

Note. The first two terms add up to 1 with good accuracy, i.e., heat capacities of dry and humid air can be considered almost the same. However, the main effect is created by the third term responsible for a phase transition, and with an increase of

T the value of a will rise faster, so that at one point it will be possible to altogether neglect heat input for heating of gas compared with heat input for evaporation of *liquid*.

Grading system

The use of the law of heat conduction 1
The calculation of T_3 1
The numerical answer for φ 1
The expression for ΔQ_1 1
The calculation of P and P_a 1
The expression for $\Delta Q'_2$ 1
The expression for Δm 1
The calculation of a 1
The expression for $\Delta Q''_2$ 1
The numerical answer for x 1