REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2025" (mathematics)

Second day

Yakutsk 2025

The booklet contains the problems of XXXII International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by A. S. Golovanov, I. I. Frolov K. P. Kokhas, V. S. Kolezhuk, A. S. Kuznetsov, O. A. Tarakanov. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

5. Each of 2025 persons arranged in a circle is either a knight, who always tells the truth, or a liar, who always lies. Each of these 2025 persons said: *There are exactly three knights among my two left neighbours, my two right neighbours and me.* What is the maximum possible number of knights among these persons?

(*A. Golovanov*)

6. In a sequence (x_n), the number x_1 is positive and rational, and

$$x_{n+1} = \frac{\{nx_n\}}{n} \quad \text{for} \quad n \geqslant 1$$

($\{a\}$ denotes the fractional part of a). Prove that this sequence contains only finitely many non-zero terms and their sum is an integer.

(*V. Kolezhuk, O, Tarakanov*)

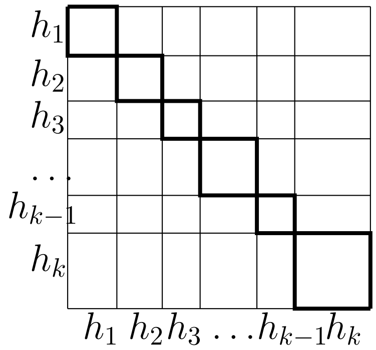
7. Each vertex of a convex quadrilateral $ABCD$ is reflected across the two sides not containing it (vertex A is reflected across BC and across CD, and so on). Prove that some 4 of 8 points thus obtained are vertices of a cyclic or circumscribed quadrilateral.

(*A. Kuznetsov*)

8. A *Latin square* (resp. *cube*) of order n is an $n\times n$ array (resp. $n\times n\times n$ array) filled with different numbers $a_1, a_2, \ldots, a_n$ occurring exactly once in each row parallel to any of its sides. Positive integers $h_1, h_2, \ldots, h_k$ are such that $n = h_1 + \ldots + h_k$. Each of the sides of a Latin square of order n is divided into parts of length $h_1, \ldots, h_k$ so that the square is cut into rectangular parts as shown in the figure. It is known that for all i the square parts $h_i \times h_i$ covering the diagonal are Latin squares of order h_i and the sets of numbers in them are disjoint.

Prove that then there exists a Latin cube of order n with similar property: its sides can be divided into parts of length $h_1, \ldots, h_k$ so that the cubes $h_i \times h_i \times h_i$ covering its diagonal are Latin cubes with disjoint sets of numbers.

(*D. Donovan, T. Kemp, J. Lefevre*)



Junior League

5. All positive real numbers are coloured with three colours (there is at least one number of each colour). Prove that there exist three numbers of three different colours so that each of them is less than the sum of the other two.

(*A. Golovanov*)

6. In a sequence (x_n), the first number x_1 is positive, and

$$x_{n+1} = \frac{\{nx_n\}}{n} \quad \text{for} \quad n \geqslant 1$$

($\{a\}$ denotes the fractional part of a). Prove that the sequence does not contain zeroes if and only if x_1 is irrational.

(*V. Kolezhuk, O, Tarakanov*)

7. There are 128 persons in each of two rooms. In one move, we can select several (disjoint) pairs of persons so that the persons in each pair are in different rooms now, and exchange persons in each pair. What minimum number of moves is needed so that every two people found themselves in different rooms at least once?

(*I. Benjamini, I. Shinkar, G. Tsur*)

8. All sides of a triangle ABC are pairwise different. Its angle bisectors AA_1, BB_1, CC_1 meet at point I. The incircle of the triangle ABC is tangent to the side BC at A_2. The circle ω_a contains A_1, A_2, and the midpoint of AI. The circles ω_b and ω_c are defined similarly. Prove that the centres of ω_a, ω_b, ω_c are collinear.

(*I. Frolov*)

SOLUTIONS

Senior League

5. The answer is 1350.

The example is given by periodic arrangement: two knights, liar, two knights, liar, etc. It is easy to see that this arrangement contains $\frac{2}{3}\cdot 2025 = 1350$ knights, and the statements of knights and liars correspond to their nature.

Bound. Suppose that the number of knights is r. Let us find the number of knights among every five consecutive persons and add all these numbers. We obtain $5r$, since every knight is counted 5 times. On the other hand, there are three knights in the five centered at a knight, and at most four in the five centered at a liar. Therefore the sum does not exceed

$$5r \leqslant 3r + 4(2025 - r).$$

Thence $r \leqslant \frac{2}{3}\cdot 2025$.

6. The hard truth about the sequence (x_n) is, this sequence being the protocol of a slow and sad decomposition of a rational number $x_1 = \frac{a}{b}$ to the form $\frac{a}{b} = s + \frac{1}{k_1} + \ldots + \frac{1}{k_m}$, where $s = [x_1]$, $k_1 > \ldots > k_m$ are positive integers, and $x_n = \frac{1}{k_i} + \ldots + \frac{1}{k_m}$ for $k_{i-1} < n \leqslant k_i$.

Indeed, $x_2 = \{x_1\}$ by the definition. Consider the moment when a new term $x_n < 1$ (not equal to x_{n-1} or simply the first if $[x_1] = 0$) appears in the sequence. Let $x_n = \frac{a}{b}$, where a and b are coprime positive integers. Consider the smallest k such that $x_n \geqslant \frac{1}{k}$. Since $x_n < \frac{1}{n-1}$ if $n > 1$, we have $k \geqslant n$. Then obviously

$$\text{all } x_i = x_n \text{ if } n \leqslant i \leqslant k. \tag{1}$$

Since $x_n < \frac{1}{k-1}$, we have $(k-1)a < b \leqslant ka < 2b$ and

$$x_{k+1} = \frac{\{\frac{ka}{b}\}}{k} = \frac{ka - b}{kb} = x_n - \frac{1}{k}. \tag{2}$$

Note that the numerator of x_{k+1} in its irreducible form is not greater than $ka - b < a$, that is, less than the numerator of x_n. As the numerator cannot decrease infinitely, at some moment the next term will be 0.

Let k_m be the index of the last non-zero term in the sequence. Then the term itself is $\frac{1}{k_m}$. If k_{m-1} is the largest index of a non-zero term different from x_{k_m}, it follows from (2) that this term is $\frac{1}{k_{m-1}} + \frac{1}{k_m}$. Restoring the sequence backwards in this way, we arrive at the desired formula.

Now we may re-write the desired sum using (1):

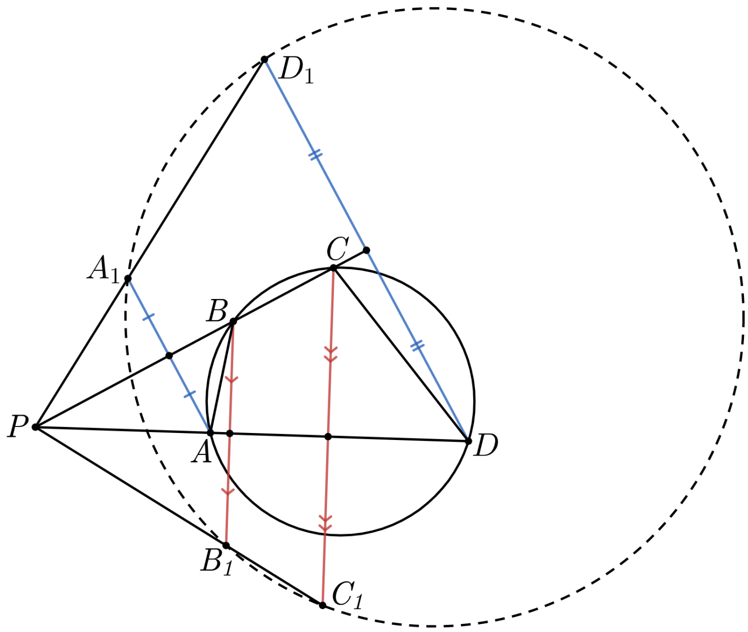
$$x_1 + x_2 + \ldots + x_{k_m} = s + k_1 \cdot \frac{1}{k_1} + k_2 \cdot \frac{1}{k_2} + \ldots + k_m \cdot \frac{1}{k_m} = s + m,$$

and this is obviously an integer.

7. If $ABCD$ is a trapezoid (or a rectangle), the claim is obvious because of axial symmetry. Let the lines BC and AD intersect at P. We consider the points A_1 and D_1 be symmetric to A and D with respect to the line BC. It follows from symmetry that P, A_1, D_1 are collinear and $PA_1 \cdot PD_1 = PA \cdot PB$.

Now let B_1 and C_1 are symmetric to B and C with respect to the line AD. It follows from symmetry that P, B_1, C_1 are collinear and $PB_1 \cdot PC_1 = PB \cdot PC$.

Thus $PA_1 \cdot PD_1 = PA \cdot PB = PB \cdot PC = PB_1 \cdot PC_1$ and therefore A_1, B_1, C_1, D_1 are concyclic (as desired) or collinear.



Note that $\angle D_1PC_1 = 3\angle BPA$ because of symmetry. Therefore in the case when A_1, B_1, C_1, D_1 are collinear, $\angle BPA$ must equal $60°$. Applying similar arguments to the common point Q of the lines AB and CD, we see that in the only remaining case we also have $\angle AQD = 60°$. In this case the quadrilateral $ABCD$ is symmetric with respect to one of its diagonals, say, BD. Then the points symmetric to C with respect to AB and AD, and the points symmetric to A with respect to BC and CD, are vertices of isosceles trapezoid, a cyclic quadrilateral.

8. Without changing the substance of the problem, we may replace a_1, a_2, …, a_n by the numbers 1, 2, …, n so that the numbers filling each $h_i \times h_i$ diagonal square are the coordinates of its rows and columns.

Let $L(r, c)$ denote the number on the intersection of the r-th row and the c-th column in the original Latin square. We will also denote by S_i set of numbers filling the $h_i \times h_i$ square on the diagonal.

We present a Latin cube as a pile of Latin squares lying in horizontal layers one above the other. In this way, a unit "cell" of a Latin cube is defined by three coordinates (r, c, ℓ), where ℓ is the number of the layer, and r, c are the numbers of the row and the column in the layer. The desired cube $\mathcal{C}$ may be defined by the following rule. Let the unit cube with coordinates (r, c, ℓ) contain the number

$$C(r, c, \ell) = L(L(r, \ell), c).$$

First we check that this rule defines a Latin cube. Indeed, let c and ℓ be fixed, and r runs from 1 to n. Then $L(r, \ell)$ also runs through the values 1, 2, …, n in order of their appearance in the ℓ-th column of the original Latin square $\mathcal{L}$. Then the second iteration $L(L(r, \ell), c)$ runs through the numbers written in the c-th column ot the square $\mathcal{L}$. In other words, we proved that in each row of unit cubes parallel to the first coordinate axis all numbers are different. It is similarly proved for the rows parallel to other axes.

It is obvious now that every cell in a unit cube (r, c, ℓ) belonging to the $h_i \times h_i \times h_i$ diagonal cube contains a number from S_i. Indeed, if the coordinates r, c, ℓ belong to S_i, then $L(r, \ell) = a \in S_i$ and $L(a, c) \in S_i$ because of our initial convention. Therefore $C(r, c, l) = L(L(r, l), c) \in S_i$.

Junior League

5. Suppose the contrary: no three numbers of different colours satisfy the conditions, that is, for every numbers x and y of different colours there are no numbers of the third colour between y and $x + y$.

Take three numbers $a < b < c$ and call their of the first, the second and the third colour respectively. We will prove by induction on k that there are no numbers of the third colour between b and $\frac{ak}{2} + b$ for each positive integer k. For $k = 2$ this is our assumption. If this is true for some k, the number $y = \frac{a(k-1)}{2} + b$ has the first or the second colour. One of the numbers a and b has other colour, and thus there are no numbers of the third colour between y and $y + a = \frac{a(k-1)}{2} + b$ or even between y and $y + b > y + a$.

On the other hand, $c < \frac{ak}{2} + b$ for some k, a contradiction.

6. If x_n is irrational, so is $x_{n+1} = \frac{\{nx_n\}}{n}$, therefore for irrational x_1 all terms of the sequence are irrational. If x_1 is rational, all numbers x_n are rational. In that case we will present each x_n as an irreducible fraction with positive integral denominator. Let us prove that for each term x_n, $n > 1$, with non-zero numerator there is a term with a smaller numerator.

Let $x_n = \frac{a}{b}$, where a and b are coprime positive integers. Consider the smallest k such that $x_n \geqslant \frac{1}{k}$. Since $x_n < \frac{1}{n-1}$ if $n > 1$, we have $k \geqslant n$.

All x_i with $n \leqslant i \leqslant k$ equal x_n. Since $x_n < \frac{1}{k-1}$, we have $(k-1)a < b \leqslant ka < 2b$ and

$$x_{k+1} = \frac{\left\{\frac{ka}{b}\right\}}{k} = \frac{ka-b}{kb}.$$

Since $ka - b < a$, the numerator of x_{k+1} is less than the numerator of x_n.

The numerator cannot decrease infinitely, therefore 0 will appear in the sequence.

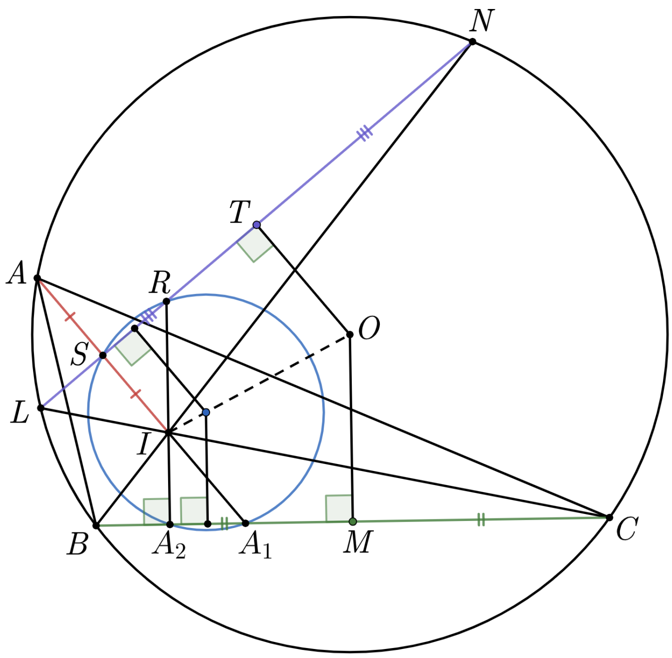
7. The answer is 7.

Bound. We use letters A and B to denote the two rooms. Suppose we attained our goal in t moves. We can give each person a paper with a string of $t+1$ letters A and B, the first letter denoting the room where she originally was, and the remaining t letters her locations after each of t moves. If two persons sat in different rooms at least once, their strings are different. This means that the number of different strings 2^{t+1} is at least $256 = 2^8$, the number of persons. Hence $t \geqslant 7$.

Example. Here is an algorithm requiring 7 moves. We give to the people all possible strings of 8 letters A и B, reserving those beginning with A to the people that were in the first room initially. In the first move the people in A with second letter B in their strings (exactly one half of all people in the room) change places with the people in B with second letter A (also one half of the room). Similarly, in i-th move each person goes to the room denoted by the $(i+1)$-th letter of her string. In this way the owners of every two different strings will find themselves in different rooms at least once.

8. Let Ω be the circumcircle of ABC. We will prove that the centres of ω_a, ω_b, ω_c belong to the line OI, where O is the centre of Ω. Let the line BB_1 and CC_1 meet Ω again at N and L, respectively.

It follows from the Incentre-excentre Lemma that $AL = LI$, $AN = NI$, therefore the line LN is the perpendicular bisector of AI. Let S be the midpoint of AI, and R the meeting point of LN and IA_2. Then S lies on the line LN and $\angle RSA_1 = \angle NSI = 90^\circ = \angle IA_2C = \angle RA_2A_1$. Thus the point R lies on the circle ω_a. Then the triangles SIR and A_2IA_1 are similar.



Since the quadrilateral $BLNC$ is cyclic, the triangles LIN and BIC are also similar. The segments SI and IA_2 are altitudes in these triangles. Therefore S and A_2 are corresponding points in these triangles. From the similarity of the triangles SIR and A_2IA_1 it follows, however, that R and A_1 are also corresponding elements in LIN и BIC. Let X be the midpoint of SR and Y the midpoint of A_2A_1. Then X and Y are also corresponding under the similarity of LIN and BIC.

Let T be the perpendicular bisector of LN and M the perpendicular bisector of BC. Then T and N are (again!) corresponding points under the similarity of LIN and BIC. Since the ratio of corresponding segments in similar triangles is constant,

$$\frac{SX}{XT} = \frac{A_2Y}{YM}. \tag{3}$$

The perpendiculars to ST at S, X, and T intersect the line IO at I, O_1, and O respectively. Thales's intercept theorem gives

$$\frac{SX}{XT} = \frac{IO_1}{O_1O}. \tag{4}$$

The perpendiculars to A_2M at A_2, Y, and M intersect the line IO at I, O_2, and O respectively. Thales's intercept theorem gives

$$\frac{A_2Y}{YM} = \frac{IO_2}{O_2O}. \tag{5}$$

Combining the results (3), (4), and (5), we obtain

$$\frac{IO_1}{O_1O} = \frac{IO_2}{O_2O}.$$

Thus $O_1 = O_2$. In other words, perpendicular bisectors of SR and A_2A_1 meet on IO and do not coincide because the triangle ABC is scalene. On the other hand, perpendicular bisectors of SR and A_2A_1 meet at the centre of the circle ω_a. Hence the centre of ω_a lies on the line IO, q.e.d.