REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2025" (mathematics)

First day

Yakutsk 2025

The booklet contains the problems of XXXII International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical commission of Russian mathematical olympiad. The booklet was compiled by M. A. Antipov, S. L. Berlov, E. I Galahova, A. S. Golovanov, K. P. Kokhas. I. A. Kukharchuk, A. S. Kuznetsov. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

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Senior league

1. To each point P in space a real number $f(P)$ is assigned so that

$$f(A) + f(B) = f(C) + f(D)$$

for every regular tetrahedron $ABCD$ with side length 1. Prove that the function f is constant.

(*A. Golovanov*)

2. Circles ω_1 and ω_2 pass through a point A and touch a line ℓ at B_1 and B_2, respectively. A variable line m passes through A and meets ω_1 and ω_2 again at (variable) points P_1 and P_2, respectively. The rays P_1B_1 and P_2B_2 intersect at a point P. Prove that the tangent to the circumcircle of PP_1P_2 at P passes through a point independent of the choice of m.

(*A. Kuznetsov*)

3. There are $n > 3$ sportsmen in a tennis club. Last month every two of them played at most one match. It is known that if A defeated B and B defeated C, then C defeated A. Prove that it is possible to select at least $n/3$ sportsmen so that no two selected sportsmen played last month.

(*S. Berlov*)

4. Positive numbers $a_1, \ldots, a_n$ and real numbers $b_1, \ldots, b_n$, c, d satisfy

$$[a_1x + b_1] + \ldots + [a_nx + b_n] = [cx + d]$$

for all real x. Prove that $a_1, \ldots, a_n$ cannot be all distinct.

(*A. Golovanov*)

Junior League

1. Positive integers a and b are given. A segment of positive integers contains more multiples of a than multiples of b. Prove that it contains at least as many multiples of $2a$ as multiples of $2b$.

(*A. Golovanov*)

2. A quadrilateral $ABCD$ is given. Its diagonals are equal and intersect at a point P. The rays AB and DC intersect at Q. It is known that $\angle BPC = 2\angle BQC$. Prove that the circle passing through B and tangent to the line AC at A is equal to the circle passing through C and tangent to the line BD at D.

(*E. Galahova, I. Kukharchuk*)

3. In a zoo shop, four aviaries arranged in a circle contain 222 parrots each. Sometimes, a zoologist takes one parrot from an aviary and lets it go; together with it, she lets go either one parrot from the opposite aviary, or two parrots from the aviary on the left, or three parrots from the aviary on the right. At some moment, only one aviary still contains parrots. What is the least possible number of the remaining parrots?

(*M. Antipov*)

4. See problem 3 of Senior league.

SOLUTIONS

Senior League

1. Consider an arbitrary regular tetrahedron $ABCD$ with side length 1. We have

$$f(A) + f(B) = f(C) + f(D).$$

Exchanging the vertices B and C we also get

$$f(A) + f(C) = f(B) + f(D).$$

Comparing the equations we see that $f(A) = f(D)$. In other words, the value of the function does not change when the point is moved by any vector of length 1.

Since every two points in the space can be connected by a polygonal chain with segments of length 1, the values of f at every two points are equal, q.e.d.

2. Let the point A' is symmetric to A with respect to the line ℓ. We will prove that all the tangents in question pass through A'. Note that

$$\angle B_1PB_2 = \angle P_1PP_2 = 180° - \angle AP_1B_1 - \angle AP_2B_2 = \\ = 180° - \angle AB_1B_2 - \angle AB_2B_1 = \angle B_1AB_2 = \angle B_1A'B_2.$$

From these equalities we obtain $\angle B_1PB_2 = \angle B_1A'B_2$. Therefore the quadrilateral $B_1A'PB_2$ is cyclic and $\angle A'PB_1 = \angle A'B_2B_1$. It follows that $\angle P_1P_2P = \angle AB_2B_1 = \angle A'B_2B_1 = \angle A'PB_1$. This means that the angle $A'PB_1$ equals P_1P_2P independently of the choice of m. This PA' is always tangent to (PP_1P_2).

3. We consider a directed graph G where vertices are sportsmen and an arc AB means that the player A defeated the player B. We will need the following claim.

Claim. The vertices of G can be coloured with three colours so that from the vertices of the first colour arrows lead only to the vertices of the second colour, from the vertices of the second colour only to the vertices of the third colour, and from the vertices of the third colour only to the vertices of the first colour.

Proof. We will prove it by induction on the number of vertices. The base case is provided by the graphs where a vertex can have only incoming arcs or only outgoing ones. In this case we use the first colour for all the tails and the second colour for all the heads.

Induction step. Suppose the graph contains a vertex B with outgoing arcs to the vertices C_1, C_2, … C_k and incoming arcs from the vertices A_1, A_2, … A_ℓ. The condition means that there is an arc from each C_j to each A_i.

Removing B from the graph we obtain a graph satisfying the condition and therefore admitting a desired colouring. Without loss of generality, assume that C_1 is of the first colour. Then all the heads of arcs going from C_1, and in particular all A_i, are of the second colour. Similarly, all the tails of the arcs going to A_1, and in particular all C_j, are of the first colour. Now we can use the third colour for B, and the claim is proved.

Returning to the problem, we apply the Claim to the graph G. In the resulting colouring, there is a colour used for at least $n/3$ vertices. As these vertices are not connected by arcs, we get the desired set.

4. Obviously $c \neq 0$. Putting $y = cx + d$ in the original equation, we obtain a similar equation with the numbers a_1/c, …, a_n/c instead of a_1, …, a_n. If some of the former are equal, so are some of the latter, therefore we can and will assume $c = 1$ and $d = 0$.

Every term $[a_s x + b_s]$ on the left is non-decreasing as a function of x, and increases by 1 when x belongs to some arithmetical progression with difference $d_s = 1/a_s$. The right-hand side increases by 1 at all integer points. This means that the above-mentioned differences d_s are positive integers, the progressions are disjoint, and their union is the set of all integers. To prove that the differences of these progressions cannot be all distinct we use the following

Lemma. Let 1, ε, ε^2, …, ε^{m-1} be all complex roots of unity of degree m. Then the sum of their k-th degrees is 0 when $k < m$ and non-zero when $k = m$.

Proof. For $k < m$ we have

$$1^k + \varepsilon^k + \varepsilon^{2k} + \ldots + \varepsilon^{(m-1)k} = \frac{\varepsilon^{mk} - 1}{\varepsilon^k - 1} = 0.$$

For $k = m$ all the terms of the sum equal 1.

We will prove that the greatest among the differences d_s appears at least twice among them. Suppose the contrary; let $d_1 > d_s$ for $2 \leqslant s \leqslant n$. Let M be the least common multiple of all d_s and ε a primitive root of unity of degree M (that is, $\varepsilon^M = 1$ and $\varepsilon^t \neq 1$ for $0 < t < M$). Every integer from 0 to $M-1$ belongs to exactly one progression. Thus all the roots of unity ε^0, ε^1, ..., ε^{M-1} of degree M are distributed among n *groups*, the s-th group being the set of all numbers ε^a, where a belongs to the s-th progression. Such group has the form

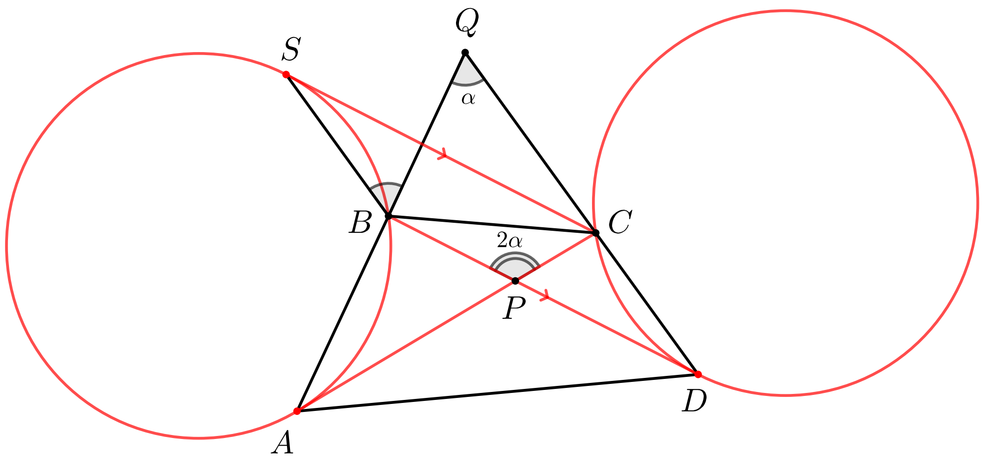
$$\varepsilon^{r_s}, \quad \varepsilon^{r_s+d_s}, \quad \ldots, \quad \varepsilon^{r_s+M-d_s}$$

and consists of all the roots of unity of degree M/d_s, multiplied by ε^{r_s}. Let us raise all the M-th roots of unity to the power of $k = \frac{M}{d_1}$. According to the Lemma, the sum of all these k-th powers is 0. On the other hand, in every group except the first one the sum of k-th powers is 0 (as $\frac{M}{d_s} > k = \frac{M}{d_1}$ for $s > 1$), and in the first group it is non-zero, a contradiction.

Junior League

1. Let the segment contain m multiples of a and n multiples of b, where $m \geq n+1$. Then it contains at least $[\frac{n}{2}]$ multiples of $2a$ and at most $[\frac{m}{2}+1]$ multiples of $2b$. Clearly the first amount is not less than the second one.

2. Define the point S so that $BDCS$ is a parallelogram. Let $\angle BQC = \alpha$, then $\angle SBQ = \alpha$ (the angles SBQ and BQC are alternate), $\angle ACS = 180^\circ - 2\alpha$ ($\angle BPC$ and $\angle ACS$ are consecutive interior angles). It follows that B belongs to the circle ω tangent to the sides of the angle ACS at A and S. The central symmetry across the centre of parallelogram $BDCS$ maps the line CS to BD, the point B to C, and the circle ω to the circle tangent to BD at D and passing through C.



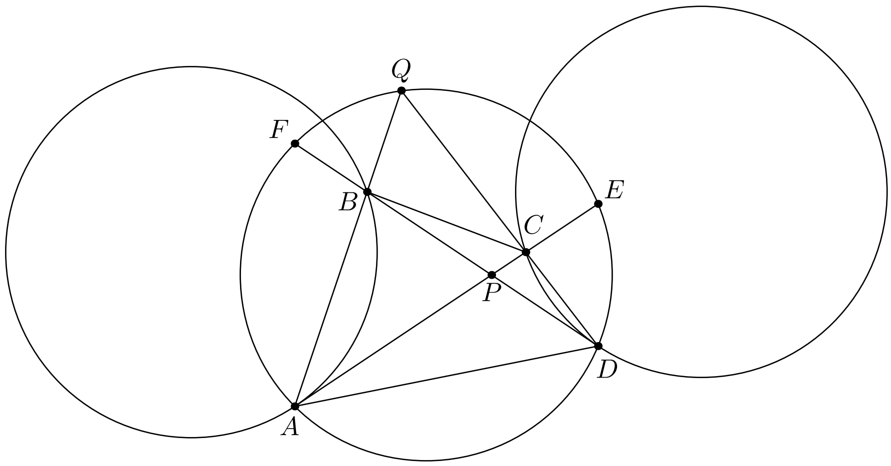
S e c o n d s o l u t i o n. Note that the circles (ACQ) and (QBD) are equal, since their chords AC and BD are equal and these chords subtend angles equal to α from the point Q. The ratio of the radii of circles (ABS) and (ACQ) is $\frac{AB}{QC}$, since from the common point A of these two circles the chords AB and QC subtend the same angle $\angle QAC$. Similarly, the ratio of the radii of circles (CDT) and (QBD) is $\frac{CD}{QB}$. To complete the proof it is enough to check that these ratios are equal, i. e.

$$AB \cdot BQ = QC \cdot CD.$$

Let E and F be the second points of intersection of the rays DP and AC with the circle (QAD), respectively. The internal angle APD equals 2α and is subtended by the arc AD equal to α, and, at the same time, by EF. Thus the arc EF also equals α, the chords AD and EF are equal, and so are the chords DF and AE. Since the segments DB and AC are equal, we infer that $BF = CE$. Applying the secant theorem, we get

$$AB \cdot BQ = BF \cdot BD = CE \cdot AC = QC \cdot CD,$$

q.e.d.



3. The answer is 5.

Let the aviaries in counter-clockwise order be A, B, C, D, and a, b, c, d the current number of parrots in these aviaries, respectively.

Example. We can perform 71 operations of the form $(a-1, b-3)$, 6 operations $(a-1, d-2)$, 70 operations $(c-1, d-3)$, and 3 operations $(c-1, b-2)$. After that all the parrots in B and D cease to be, and A contains 5 more parrots than C. After that using the operation $(a-1, c-1)$ all the others join the choir invisible, except some five unfortunates in A.

Bound. Note that что $3^4 \equiv 1$, $3^2+1 \equiv 0$, and $2 \cdot 3^3 + 1 \equiv 0 \pmod 5$. Therefore the sum

$$a + 3b + 3^2c + 3^3d \mod 5$$

never changes under the operation. Initially this sum equals 0 mod 5, so in the end, if all the parrots in B, C, and D are ex-parrots, the number of parrots in A should be divisible by 5.

4. See Problem 3, Senior league.