REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2024" (mathematics)

Second day

Yakutsk 2024

The booklet contains the problems of XXXI International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by S. L. Berlov, A. S. Golovanov, K. P. Kokhas, Yu. V. Kuzmenko, A. A. Mardanov, F. V. Petrov. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

5. In a 25×25 chessboard some squares are marked so that in each subboard of size 13×13 or 4×4 at least half the squares are marked. What is the minimum possible number of marked squares in the entire chessboard? (*S. Berlov*)

6. A triangle ABC is given. Variable point X runs through the arc of its circumcircle which does not contain A. Points Y on the ray XB and Z on the ray XC are defined so that $XA = XY = XZ$. Prove that line YZ goes through a fixed point. (*A. Kuznetsov*)

7. Given are two polynomials f and g of degree 100 with real coefficients. For each positive integer n there is an integer k such that

$$\frac{f(k)}{g(k)} = \frac{n+1}{n}.$$

Prove that f and g have a common non-constant factor. (*A. Golovanov*)

8. A graph G has n vertices ($n > 1$). For each edge e let $c(e)$ be the number of vertices of the largest complete subgraph containing e. Prove the inequality (the summation is over all edges of G):

$$\sum_e \frac{c(e)}{c(e)-1} \leqslant \frac{n^2}{2}.$$

(*D. Malec, C. Tompkins*)

Junior League

5. In a 25×25 chessboard some squares are marked so that in each subboard of size 13×13 or 4×4 at least half the squares are marked. What is the minimum possible number of marked squares in the entire chessboard? (*S. Berlov*)

6. Extension of angle bisector BL of the triangle ABC (where $AB < BC$) meets its circumcircle at N. Let M be the midpoint of BL. Isosceles triangle BDC with base BC and angle equal to ABC at D is constructed outside the triangle ABC. Prove that $CM \perp DN$. (*A. Mardanov*)

7. Given are quadratic trinomials f and g with integral coefficients. For each positive integer n there is an integer k such that

$$\frac{f(k)}{g(k)} = \frac{n+1}{n}.$$

Prove that f and g have a common root. (*A. Golovanov*)

8. A toy factory produces several kinds of clay toys. The toys are painted in k colours. *Diversity* of a colour is the number of *different* toys of that colour. (Thus, if there are 5 blue cats, 7 blue mice and nothing else is blue, the diversity of colour blue is 2.) The painting protocol requires that *each colour is used and the diversities of each two colours are different.* The toys in the store could be painted according to the protocol. However, a batch of clay Cheburashkas arrived at the store before painting (there were no Cheburashkas before). The number of Cheburashkas is not less that the number of the toys of any other kind. The total number of all toys, including Cheburashkas, is at least $\frac{(k+1)(k+2)}{2}$. Prove that now the toys can be painted in $k+1$ colours according to the protocol. (*F. Petrov*)

SOLUTIONS

Senior League

5. Answer: 313, that is, a little more than one half.

Lower bound. Consider two 13×13 subboards containing opposite angles of the original board. They have one common square, therefore their union contains at least $85 + 85 - 1 = 169$ marked squares. The two remaining 12×12 subboards can be divided into 4×4 subboards and therefore contain at least 144 marked squares, which gives at least 313 marked squares on the whole board.

Example. In the chess colouring of the original 25×25 chessboard with black angles we mark all white squares (312) and one central (black) square. Then obviously at least half of the squares is marked in each 4×4 subboard. In each 13×13 subboard it is also true since every such board contains 84 or 85 white squares and the central square.

6. We show that YZ contains the incentre I of triangle ABC. Since X lies on the arc BC of the circumcircle of ABC, $\angle BXA = \angle BCA$. Triangle AXY is isosceles, therefore

$$\angle AYX = 90^\circ - \frac{1}{2}\angle YXA = 90^\circ - \frac{1}{2}\angle BCA.$$

Angle $\angle AIB$ between bisectors is $90^\circ + \frac{1}{2}\angle BCA$, hence the quadrilateral $AIBY$ is cyclic. Thus, $\angle AIY = \angle ABY$. Similarly, $\angle AIZ = \angle ACZ$. As the quadrilateral $ABXC$ is cyclic, we have $180^\circ = \angle ABY + \angle ACZ = \angle AIY + \angle AIZ$. It follows that Y, I, Z are collinear.

7. Consider the difference $f_1(x) = f(x) - g(x)$. For each positive integer n there is an integer k_n such that $\frac{f_1(k_n)}{g(k_n)} = \frac{1}{n}$. Let us divide $g(x)$ by $f_1(x)$ with remainder:

$$g(x) = f_1(x)q(x) + r(x).$$

Dividing this equation by $g(x)$ and substituting all k_n we find that

$$q(k_n) + \frac{r(k_n)}{f_1(k_n)} = n \qquad (*)$$

for all n. Obviously $\lim_{n\to\infty} k_n = \infty$ (since k_n is a sequence of different integers), and $\lim_{n\to\infty} \frac{r(k_n)}{f_1(k_n)} = 0$, that is for all n except possibly finite number of values, n is the nearest integer to $q(k_n)$.

The polynomial $q(x)$ is thus unbounded and therefore for some C it is monotonous for $x > C$ and for $x < -C$, and never attains at $|x| \leqslant C$ any values attained at $|x| > C$.

If the degree of $q(x)$ is greater than 1, the inequality $|q(x+1)-q(x)| > 3$ holds for large enough x. This means that among three consecutive integers greater than C at most two can be nearest integers to values of $q(x)$ at integral points, a contradiction. Therefore $q(x)$ is linear: $q(x) = ax + b$.

It follows from $(*)$ that the difference between $ak_n + b$ and n becomes arbitrarily large for large enough n, and, consequently, the difference $a(k_{n+1} - k_n)$ becomes arbitrarily close to 1. This difference, however, is a multiplied by an integer, and therefore equals 1 for large enough n. Thus the difference $k_n a + b - n$ is eventually constant and tends to 0, that is, equals 0 for large enough n.

We have proved that $r(n) = 0$ for all large enough n, hence r is identically 0, and $g(x) = (ax+b)f_1(x)$, i. e. f_1 divides g and therefore $f = g + f_1$, q.e.d.

R e m a r k. We have seen that f and g have a common factor of degree 99. Since this degree is odd, the common factor has a real root, which is also a common root of f and g.

8. In the following, $|X|$ denotes the number of elements of a set X.

We prove the claim by induction on n. Let G be an n-vertex graph with edge set E. If the graph contains no edges, then the bound follows trivially so assume G has at least one edge. Let k be the size of the largest clique in G, and let C be a clique of size k. We split the edges E into three parts: E_C, the edges within C; $E_{G\setminus C}$, the edges within $G \setminus C$; and E_S, the edges that connect C and $G \setminus C$. We bound the contribution to the sum from each of these separately.

1. Since C is a clique of maximum size in G, we know that E_C contains $k(k-1)/2$ edges, all with $c(e) = k$. So we can see that

$$\sum_{e \in E_C} \frac{c(e)}{c(e)-1} = \frac{k(k-1)}{2} \cdot \frac{k}{k-1} = \frac{k^2}{2}.$$

2. For all $v \in V(G \setminus C)$, let $C_v = \{w \in C \mid \{v, w\} \in E\}$. Note that $C_v \cup \{v\}$ is itself a clique, and so we may conclude that $|C_v| + 1 \leqslant k$, and

$c(e) \geqslant |C_v| + 1$ for all edges e that connect v to C. Thus, we obtain that

$$\sum_{e\in E_S} \frac{c(e)}{c(e)-1} \leqslant \sum_{v\in V(G\setminus C)} |C_v| \cdot \frac{|C_v|+1}{|C_v|} \leqslant \sum_{v\in V(G\setminus C)} k = (n-k)k.$$

3. Lastly, our induction hypothesis implies that

$$\sum_{e\in E_{G\setminus C}} \frac{c(e)}{c(e)-1} \leqslant \frac{(n-k)^2}{2}.$$

Combining all three of the estimates above we obtain that

$$\sum_{e\in E} \frac{c(e)}{c(e)-1} \leqslant \frac{k^2}{2} + k(n-k) + \frac{(n-k)^2}{2} = \frac{n^2}{2},$$

as required.

R e m a r k. The equality holds if and only if G is a multipartite graph with classes of equal size.

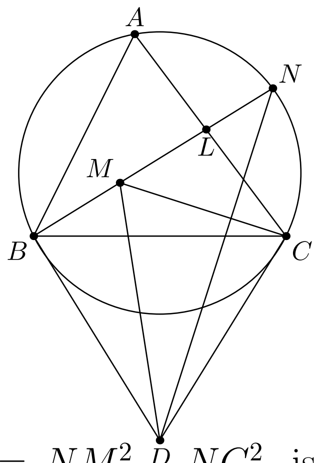
Junior League

5. See Problem 5, Senior league.

6. Note that $BL \perp BD$, since in isosceles triangle BCD

$$\angle CBD = 90^\circ - \frac{1}{2}\angle BDC = 90^\circ - \frac{1}{2}\angle ABC = 90^\circ - \angle LBC.$$

Then



$$\begin{aligned} DM^2 - DC^2 = BM^2 = \quad & (1)\\ = NM^2 - (NM - BM)(NM + BM) = \quad & (2)\\ = NM^2 - NL \cdot NB = \quad & (3)\\ = NM^2 - NC^2 \quad & (4) \end{aligned}$$

Here (1) is Pythagorean theorem, and equality of (1) and (2) is checked by expanding.

It is well-known that the equation $DM^2 - DC^2 = NM^2 - NC^2$, is equivalent to the desired perpendicularity.

7. Consider the difference $f_1(x) = f(x) - g(x)$. For each positive integer n there is an integer k_n such that $\frac{f_1(k_n)}{g(k_n)} = \frac{1}{n}$. In other words, $\frac{f_1(x)}{g(x)}$ can be arbitrarily small for arbitrarily large x. Therefore the polynomial $f_1(x)$ is linear (and non-constant, otherwise quadratic trinomial g/f_1 attains all positive integral values, which is impossible).

Let us divide $g(x)$ by $f_1(x)$ with remainder:

$$g(x) = f_1(x)q(x) + r(x).$$

Since f_1 is linear and the coefficients of polynomials are integers, the remainder r is a rational number (a constant polynomial). The partial quotient $q(x) = ax + b$ is also linear, and for each positive integer n (except possibly the unique root of f_1) there is an integer k_n such that $q(k_n) + \frac{r}{k_n} = n$. It follows that the coefficients of $q(x)$ are also rational, and the numbers $q(k_n)$ are common fractions with bounded denominators (their denominators divide the common denominator of the coefficients of $q(x)$). Then the denominators of the numbers $\frac{r}{k_n}$ are also bounded, which implies $r = 0$. Therefore the polynomial $f_1(x)$ divides $g(x)$ and consequently $f(x) = g(x) + f_1(x)$, so the root of f_1 is also common root of f and g.

8. We prove the claim by induction on k.

The base case, $k = 0$: no toys in the store, there is nothing to paint, and now they give us Cheburashkas and say: "Paint it in one colour". OK, done.

Induction step. Let us reduce the problem of painting toys with $k+1$ colours to smaller number of colours. Informally speaking, our main concern in this reduction will be the number of Cheburashkas (they have to retain the majority).

We begin with a general observation from the viewpoint of the painter (that is, the person responsible for painting). Let us arrange the colours in increasing order of diversity. Then the diversity of i-th colour is at least i. The painter can minimize his or her efforts, making the diversity of i-th colour exactly i for each i, moreover, painting only one toy of each kind appearing in i-th colour. With such work ethics, some toys may remain unpainted. The painter can postpone painting these toys to the last minute, and even then paint them with any available colours – this will not breach the protocol.

To each kind of toys we will allot a separate shelf with places numbered from left to right. On each shelf we reserve i–th place for the toy of i–th colour (if available). Then i–th "column" of our closet will contain exactly i toys, and the first k columns $\frac{k(k+1)}{2}$ toys. We will put unpainted toys on the right end of the appropriate shelves. We will also have a special shelf for Cheburashkas. After the delivery of Cheburashkas we put them in the "unpainted" part of the shelf, but consider them so far as a separate group, distinct from the unpainted toys.

The painter hidden in each of us suggests that to paint the toys with $k+1$ colour according to protocol it is enough to leave only $\frac{(k+1)(k+2)}{2}$ in

Species	1	2	3	4	...	$k-1$	k	unpainted
Hares	$\bullet$	$\bullet$		$\bullet$	...	...	...	$\circ$ $\circ$ $\circ$
Elephants			$\bullet$	$\bullet$	...	...	...	$\circ$ $\circ$
Edgehogs		$\bullet$	$\bullet$		...	...	...	$\circ$
Tiggers			$\bullet$					
Elks				$\bullet$				
Owls				$\bullet$				
...								
Cheburashkas								$\circ$ $\circ$ $\circ$ $\circ$ ...

Figure 1: Arranging toys on shelves

the closet, all painted toys and appropriate number of Cheburashkas, plus some unpainted toys, if we don't have enough Cheburashkas. Assume that Fig. 1. shows exactly that arrangement of toys.

Note that, because of the nature of our closet, each shelf contains at most k painted toys. And since we have left $\frac{(k+1)(k+2)}{2}$ toys, including $\frac{k(k+1)}{2} = \frac{(k+1)(k+2)}{2} - k$ painted toys, Cheburashkas still form the most numerous (not strictly) kind of toys, even if some of them are outside the closet.

We have to paint the toys in the closet with $k+1$ colours. Let us try to solve the most complicated problem, that of choosing the toys for $(k+1)$-th colour. Plotting to apply induction, we reserve for $(k+1)$-th colour the toys now coloured with k-th colour; for that we just rename k-th place of each shelf to $(k+1)$-th. Now each shelf has places from 1 to $(k-1)$, the place with number $k+1$ and the part for unpainted toys. Already we have k toys to paint with $(k+1)$-th colour, and we need one more.

First of all, if some shelf contains an unpainted toy but no toy assigned to painting with $(k+1)$-th colour, assign it, too, to painting with $(k+1)$-th colour. Then the list of toys to paint with the $(k+1)$-th colour is complete, there are $\frac{k(k+1)}{2}$ toys outside that list, these toys satisfy the diversity rule for $k-1$ colours, and Cheburashkas are still the most numerous group among them. Therefore these toys can be painted with k colours by the induction hypothesis.

Now suppose that there are some unpainted toys, but none of them can be transferred to $(k+1)$-th colour. Then each shelf containing (at least one) unpainted toy contains also an empty place among places from 1 to $k-1$. Otherwise, every such shelf contains at least k toys before the arrival of Cheburaskas, therefore, at least k Cheburashkas arrived. In view of the above artistic observations, it means that there are no unpainted toys at all, that is, there are $\frac{k(k+1)}{2}$ painted toys and k Cheburashkas. This means that the case is impossible.

Let us try to repair the colouring we have. First we rearrange the shelves so that the shelves assigned to painting with $(k+1)$-th colour are at the top (see. Fig. 2). These shelves contain all unpainted toys.

Species	1	2	3	4	...	$k-1$	$k+1$	unpainted
Hares	•	•	×	•	...	...	•	○ ○ ○
Elephants			•	•	...	...	•	○ ○
Edgehogs		•	•		...	...	•	○
...	...					...	...	...
Tiggers			$\underline{\bullet}$					
Elks				•				
Owls				•				
...								
Cheburashkas								○ ○ ○ ○ ...

Figure 2: Painting with $(k+1)$-th colour.

C a s e 1. One of the upper shelves contains an unpainted toy and empty i-th place (that is, it dos not contain a toy of i-th colour), but some of the bottom shelves has a toy in the i-th place (in the figure 2 we marked the empty space in the third column of the row "hares" and underlined the occupied place in the row "tiggers"). Now we rearrange the colouring: we paint the unpainted toy from the top shelf with i-th colour, and the toy of i-th colour from the bottom shelf will be discoloured and assigned to $(k+1)$-th colour (Fig. 3). After that we can apply the induction hypothesis.

Species	1	2	3	4	...	$k-1$	$k+1$	unpainted
Hares	•	•	•	•	...	...	•	○ ○
Elephants			•	•	...	...	•	○ ○
Edgehogs		•	•		...	...	•	○
...	...					...	...	...
Tiggers			_				•	
Elks				•				
Owls				•				
...								
Cheburashkas								○ ○ ○ ○ ...

Figure 3: Painting with $(k+1)$-th colour.

C a s e 2 (opposite to Case 1). For each empty place on each top shelf all places under it on the bottom shelves are occupied. This means that the most numerous kind of toys after Cheburashkas is in the upper part of the closet. Then we assign one Cheburashka to painting with $(k+1)$-th colour. Since each row in the upper part contains a toy assigned to colour $k+1$, after removing all toys assigned to that colour we get a situation where the induction hypothesis holds (in terms of diversity and dominating of Cheburashkas).