REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2024" (mathematics)

First day

Yakutsk 2024

The booklet contains the problems of XXXI International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical commission of Russian mathematical olympiad. The booklet was compiled by S. L. Berlov, A. S. Golovanov, K. P. Kokhas. Yu. V. Kuzmenko, F. V. Petrov, A. D. Tereshin, I. I. Frolov, D. Yu. Shiryaev. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

1. Prove that a positive integer of the form n^4+1 can have more than 1000 divisors of the form a^4+1 with integral a. *(A. Golovanov)*

2. Chip and Dale play on a 100×100 table. In the beginning, a chess king stands in the upper left corner of the table. At each move the king is moved one square right, down or right-down diagonally. A player cannot move in the direction used by his opponent in the previous move. The players move in turn, Chip begins. The player that cannot move loses. Which player has a winning strategy?
(D. Shiryaev, O. Badazhkova)

3. All perfect squares, and all perfect squares multiplied by two, are written in a row in increasing order. Let $f(n)$ be the n-th number in this sequence. (For instance, $f(1)=1$, $f(2)=2$, $f(3)=4$, $f(4)=8$.) Is there an integer n such that all the numbers

$$f(n), \quad f(2n), \quad f(3n), \quad \ldots, \quad f(10n^2)$$

are perfect squares? *(F. Petrov)*

4. A triangle ABC is given. The segment connecting the points where the excircles touch AB and AC meets the bisector of angle C at X. The segment connecting the points where the excircles touch BC and AC meets the bisector of angle A at Y. Prove that the midpoint of XY is equidistant from A and C. *(I. Frolov)*

Junior League

1. *Triangular numbers* are numbers of the form $1+2+\ldots+n$ with positive integral n, i. e. 1, 3, 6, 10, ... Determine the greatest non-triangular integer which is not a sum of different triangular numbers. *(A. Golovanov)*

2. We call *an edgehog* a graph in which one vertex is connected with all others and there are no other edges; the number of vertices of this graph we call *the size* of the edgehog. A graph G with $n > 1$ vertices is given. For each edge e, let us denote $s(e)$ the size of the largest edgehog in G containing this edge. Prove the inequality (the summation is over all edges of G):

$$\sum_e \frac{1}{s(e)} \leqslant \frac{n}{2}.$$

(D. Malec, C. Tompkins)

3. Three sportsmen ran with different constant velocities along a track of length 1. They started simultaneously at one end of the track. Upon reaching an end of the track, every sportsman immediately turned around and continued running in the opposite direction. After some time all three sportsmen met at the starting position and finished the practice. What is the maximum S for which it is certainly true that at some moment the sum of pairwise distances between the sportsmen was at least S?
(A. Golovanov, I. Rubanov)

4. A triangle ABC is given. N and M are the midpoints of AB and BC, respectively. The bisector of angle B meets the segment MN at E. H is the base of the altitude drawn from B in the triangle ABC. The point T on the circumcircle of ABC is such that the circumcircles of TMN and ABC are tangent. Prove that points T, H, E, B are concyclic. *(M. Yumatov)*

SOLUTIONS

Senior League

1. First solution. For each integer a there is an integer $b > a$ such that a^4+1 divides b^4+1). Indeed, if $b \equiv a \pmod{a^4+1}$ (for instance, $b = a^4 + a + 1$), then $b^4 \equiv a^4 \equiv -1 \pmod{a^4+1}$.

Using that observation we can construct an increasing sequence a_0, a_1, ... such that $a_i^4 + 1$ divides $a_{i+1}^4 + 1$ for all i. Then obviously $a_n^4 + 1$ has at least n divisors of the desired form.

Second solution. Let us take an arbitrary $a > 1$ and construct the sequence (a_k) defined by

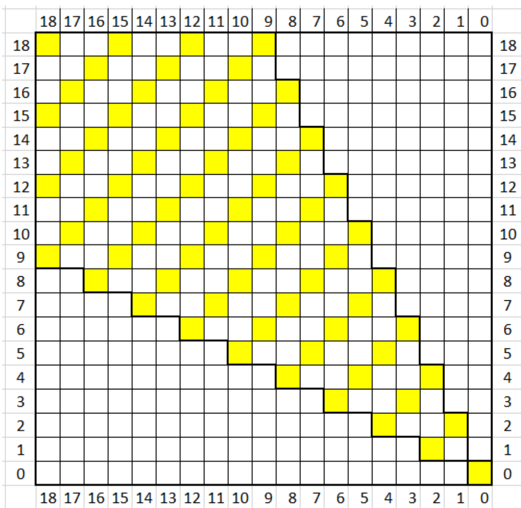
$$a_1 = a^4 + 1, \qquad a_{k+1} = (a_1 a_2 \cdots a_k)^4 + 1, \quad k = 1, 2, \ldots, 1000.$$

In this sequence a_i and a_j are coprime for $i \neq j$, and for each a_i the congruence $x^4 \equiv -1 \pmod{a_i}$ has a solution, say x_i. Now, by Chinese Remainder Theorem, we can find a number n satisfying the system of congruences

$$n \equiv x_i \pmod{a_i}, \qquad i = 1, 2, \ldots, 1001.$$

Then n has the desired property.

2. Answer: Dale wins.



We define the position of a square by coordinates (a, b), where a is the number of column between 99 and 0, b is the number of row betweenn 99 and 0 and the starting point is $(99, 99)$. We call a square (a, b) *good* if 3 divides $a + b$, $a \leqslant 2b$, and $b \leqslant 2a$; in particular, the squares $(0, 0)$ and $(99, 99)$ are good. Good squares are well described as squares accessible from $(0, 0)$ by knight's moves of to the left and to the top.

Here is a winning strategy for Dale. At every move Chip leaves a good square and Dale will try to get back to a good square. To do it, Dale should respond to a diagonal move by vertical or horizontal one (drawing nearer to the diagonal (x, x)), and to a vertical or horizontal move by diagonal one. If Dale can follow this strategy, he will inevitably reach the good point $(0, 0)$ and win.

The only case where Dale cannot follow that strategy arises when Chip moves right from $(a, 2a)$ to $(a-1, 2a)$ (and, similarly, when he moves from $(2a, a)$ to $(2a, a-1)$). In this case Dale has to change the strategy: from that moment he should always move down (or, respectively, right). Then the king will always stay in the region $2a < b$ (or $2b < a$), and Dale will be always able to move.

3. The answer is no.

Clearly $n = 1$ does not have the required property, so we assume $n > 1$.

Let $f(n)$ be a perfect square, and the first n numbers in the row contain u squares and v squares multiplied by 2, i. e. $n = u + v$, where $2v^2 < u^2 < 2(v+1)^2$. Consider the difference $\varepsilon = u - v\sqrt{2}$. Since $v\sqrt{2} < u < (v+1)\sqrt{2}$, we have $0 < \varepsilon < \sqrt{2}$.

Case 1. Let $\varepsilon < 1$. Consider the smallest integral k such that $k\varepsilon > \sqrt{2}$; for it we obviously have $k\varepsilon < \sqrt{2} + 1$. Then

$$ku > (kv + 1)\sqrt{2} = k(u - \varepsilon) + \sqrt{2} > ku - 1,$$

that is, $(ku)^2 > 2(kv + 1)^2 > (ku - 1)^2$, therefore,

$$f(kn) = f(kv + 1 + ku - 1) = 2(kv + 1)^2$$

is not a square. However,

$$\varepsilon = u - v\sqrt{2} = \frac{u^2 - 2v^2}{u + v\sqrt{2}} \geqslant \frac{1}{u + v\sqrt{2}} > \frac{1}{\sqrt{2}(u + v)} = \frac{1}{n\sqrt{2}},$$

thus $k < 2n$, and n does not satisfy the requirement.

Case 2. Now let $\varepsilon > 1$ (clearly $\varepsilon = 1$ is impossible). Then

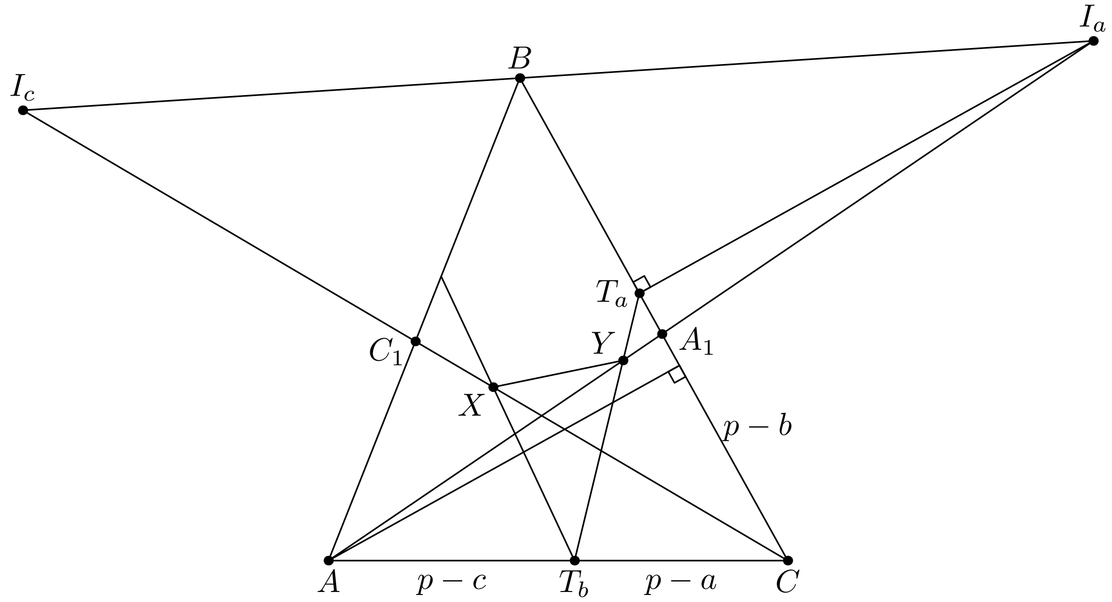
$$(2v + 1)\sqrt{2} < 2u < (2v + 2)\sqrt{2}.$$

Since $f(2n) = f(2u+2v)$ is a perfect square, $f(2n) = (2u-1)^2$, therefore, $2u - 1 > (2v + 1)\sqrt{2}$. At the same time

$$(2u - 1) - (2v + 1)\sqrt{2} = 2(u - v\sqrt{2}) - 1 - \sqrt{2} < \sqrt{2} - 1 < 1.$$

Now we can apply above argument (see case 1) to the number $2n$. That means that some of the numbers $f(2n)$, $f(4n)$, …, $f(4n^2)$ is not a perfect square.

4. Let I_a, I_c the respective circumcentres, A_1 and C_1 the bases of respective bisectors. We make use of the following theorem (Trillium lemma, or Mansion's Theorem): the midpoint of I_aI_c is equidistant from A and C, that is, lies on the perpendicular bisector of AC. To prove that the midpoints of AC, XY, and I_aI_c are collinear, we check that $AY/AI_a = CX/CI_c$.



Let a, b, c, and p denote the respective sides of triangle ABC and its semiperimeter. The points where the excircles touch the sides BC and AC we denote T_a and T_b respectively. Then $AT_b = p - c$, $CT_b = p - a$, and $CT_a = p - b$. Using the angle bisector theorem we find $CA_1 = \frac{ab}{b+c}$. Applying Menelaus's theorem to the triangle AA_1C gives

$$\frac{AY}{YA_1} = \frac{AT_b}{T_bC} \cdot \frac{CT_a}{T_aA_1} = \frac{p-c}{p-a} \cdot \frac{p-b}{p-b-\frac{ab}{b+c}} =$$

$$= \frac{a+b-c}{b+c-a} \cdot \frac{(a+c-b)(b+c)}{(a+c-b)(b+c)-2ab} = \frac{(a+b-c)(a+c-b)(b+c)}{(b+c-a)(a+b+c)(c-b)}.$$

Then

$$\frac{AY}{AA_1} = \frac{(a+b-c)(a+c-b)(b+c)}{(b+c-a)(a+b+c)(c-b)+(a+b-c)(a+c-b)(b+c)} =$$

$$= \frac{(a+b-c)(a+c-b)(b+c)}{2b(a^2+c^2-b^2)}.$$

In order to find $\frac{AA_1}{AI_a}$, we draw perpendiculars to BC from A and I_a to BC which are the altitude $h_a = \frac{2S}{a}$ and the exradius $r_a = \frac{2S}{b+c-a}$, where S is the area of ADC. It follows that

$$\frac{AA_1}{AI_a} = \frac{h_a}{h_a + r_a} = \frac{\frac{2S}{a}}{\frac{2S}{a} + \frac{2S}{b+c-a}} = \frac{b+c-a}{b+c}.$$

Multiplying the two ratios we get

$$\frac{AY}{AI_a} = \frac{AY}{AA_1} \cdot \frac{AA_1}{AI_a} = \frac{(a+b-c)(a+c-b)(b+c-a)}{2b(a^2+c^2-b^2)}.$$

This expression in symmetric with respect to b and c, therefore the ratio is the same. Thus the problem is solved.

Junior League

1. The answer is 33.

Triangular numbers not exceeding 33 are 1, 3, 6, 10, 15, 21, and 28. The sum of the first five numbers is less than 33, therefore to represent 33 we need 21 or 28. However, if 28 is used, the other terms should add up to 5, which is impossible. And if we use 21, the rest is 12, and 12 cannot be obtained either with 10 (the rest becomes 2) or without 10 ($1+3+6 < 12$).

In the following, t_n denotes the n-th triangular number $\frac{n(n+1)}{2}$.

We claim that all numbers greater than 33 can be represented in the desired form. To establish it, we prove, firstlym that all numbers between 34 and 78 can be represented using summands not exceeding 36, and, secondly (by induction on $n \geqslant 9$), that all numbers between 34 and $78+t_9+\ldots+t_n$ can be represented using summands not exceeding t_n. Since $78+t_9+\ldots+t_n$ can be arbitrarily large, this solves the problem.

All numbers between 9 and 22, except 12, are represented by summands not exceeding 15: $9 = 6+3$, $10 = 10$, $11 = 10+1$, $13 = 10+3$, $14 = 10+3+1$, $15 = 15$, $16 = 10+6$, $17 = 10+6+1$, $18 = 15+3$, $19 = 15+3+1$, $20 = 10+6+3+1$, $21 = 15+6$, $22 = 15+6+1$.

Adding 21 to these numbers, we represent all numbers between 30 and 43, except 33, by summands not exceeding 21.

Adding 28 to the numbers between 16 and 43, we obtain all numbers between 44 and 71, except 51 and 61.

Noting that $51 = 36+15+10$, $61 = 36+15+10+6+3+1$, we prove our first claim.

To prove the second claim, we add to numbers between 34 and 78, already obtained, the number $t_9 = 45$, thus obtaining numbers from 79 to $78 + t_9 = 123$.

Suppose that all numbers between 34 and $78+t_9+\ldots+t_n$ are represented by summands not exceeding t_n. Adding t_{n+1}, we get all numbers between $34+t_{n+1}$ and $78+t_9+\ldots+t_{n+1}$. It remains to note that the interval between 34 and $78+t_9+\ldots+t_n$ and the interval between $34+t_{n+1}$ $78+t_9+\ldots+t_{n+1}$ intersect, since $34+t_{n+1} \leqslant 78+t_9+\ldots+t_n$. Indeed, subtracting 34 and t_n from both sides, we get the obvious inequality $n+1 \leqslant 45+t_9+\ldots+t_{n-1}$.

2. For each vertex v let $w(v)$ be the sum of numbers $\frac{1}{s(e)}$ over all edges going from v. For every such e we obviously have $s(e) \geqslant \deg(v)$, and therefore

$$w(v) \leqslant \sum_{e:v\in e} \frac{1}{\deg v} = 1.$$

Then

$$2\sum_{e} \frac{1}{s(e)} = \sum_{v} w(v) \leqslant n,$$

as desired.

3. Answer: $\frac{8}{5}$.

Clearly the sum of pairwise distances between the three sportsmen is twice the largest of these distances. It is also obvious that the velocities of the sportsmen are proportional to three positive integers (if the sportsmen covered the track a, b, and c times during the workout, their velocities are proportional to the numbers a, b, and c).

First we consider the movement of two sportsmen.

Lemma. Let the velocities of two sportsmen are proportional to coprime integers p and $q > p$. Then the greatest distance between the two sportsmen is 1 if one of the numbers p and q is even, and $1-\frac{1}{q}$ if p and q are both odd.

Proof. If p is even and q is odd, when the first sportsman has run the distance p, and the second one the distance q, the sportsmen are in the opposite ends of the track, at a distance 1.

Let p and q be odd. Note that in the interval between two consecutive moments when one of the sportsmen is at an end of the track, both sportsmen move with constant velocities without changing direction; therefore the distance between them is the greatest in one of these two moments. If the first sportsman is at an end of the track, he has run an integral distance n. Then the second sportsman has run the distance $\frac{nq}{p}$. This number is a fraction with denominator p. It can be an integer only when p divides n, and then its parity is that of n. This means that the second sportsmen is where the first one is. In all other cases the distance between the sportsmen is less than 1 and therefore does not exceed $1-\frac{1}{p}$. Similarly, when the second sportsmen is at an end of the track, the distance between the sportsmen does not exceed $1-\frac{1}{q}$.

We claim that at some moment the distance between the sportsmen equals $1-\frac{1}{q}$. Since p and q are coprime, there is a positive integer $k \leqslant q$ satisfying the congruence

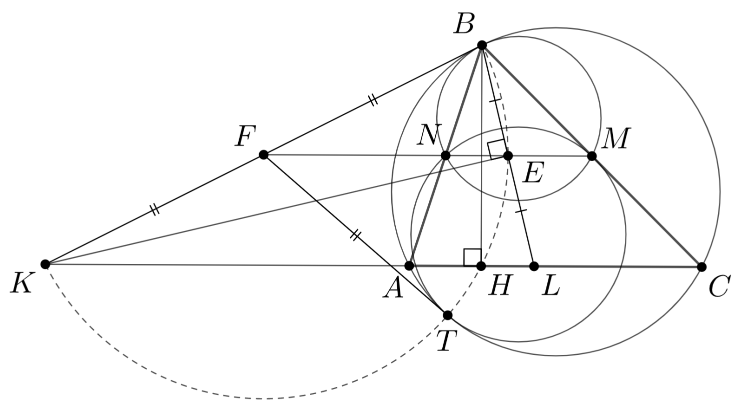
$$kp \equiv \frac{q+1}{2} \pmod{q}.$$

This congruence means that $kp = mq + \frac{q+1}{2}$, that is, $2kp = (2m+1)q+1$ for some integral m. Therefore, when the fast sportsmen has run the track k times in both directions and returned to the start, the slow sportsman has run the distance $\frac{2kp}{q} = 2m+1+\frac{1}{q}$, and is at a distance $1-\frac{1}{q}$ from the start. The lemma is proved.

If the ratio of the sportsmen's velocities is $1:3:5$, by virtue of the Lemma the greatest distance between the first two sportsmen is $\frac{2}{3}$, and between the third one and any other — $\frac{4}{5}$. Thus the greatest sum of distances between the sportsmen is $\frac{8}{5}$.

It is seen from the Lemma that the greatest distance between two sportsmen whose velocities are proportional to coprime odd numbers p and $q>p$ can be less than $\frac{4}{5}$ only when $q=3$ (and, consequently, $p=1$). But if the middle-paced sportsman is 3 times faster than the slowest one and 3 times slower than the fastest one, the ratio of other two velocities is 9, and the distance between the other two sportsmen is $\frac{8}{9} > \frac{4}{5}$ at some moment.

4. We use the notation (XYZ) for the circumcircle of triangle XYZ. Since the triangles ABC and BMN are homothetic with centre B, the circles (ABC) and (BMN) are tangent at B. Let F be the intersection point of tangents to the circle (ABC) at B and T. Note that these tangents are also tangents to the circles (BMN) and (TMN), therefore the powers of F with respect to these circles are equal. Thus F lies on the radical axis of the circles (BMN) and (TMN), that is, on the line MN.



Let L and K be the points where the line AC meets the lines BE and BF respectively. Since MN is the midline of triangle ABC, E and F are midpoints of BL and BK. Because BK is tangent to the circle (ABC), $\angle KBA = \angle BCA$, hence

$$\angle KBL = \angle KBA + \angle ABL = \angle BCA + \angle CBL = \angle KLB.$$

Thus the triangle KBL is isosceles, and its median KE is also its altitude. It remains to note that the circle with diameter BK contains all four points T, H, E, B.