REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2023" (mathematics)

Second day

Yakutsk 2023

The booklet contains the problems of XXX International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by A. S. Golovanov, K. S. Ivanov, K. P. Kokhas, F. V. Petrov. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

5. A small ship sails on an infinite coordinate sea. At the moment t the ship is at the point with coordinates $\big(f(t), g(t)\big)$, where f and g are two polynomials of third degree. Yesterday at 2 p.m. the ship was at the same point as at 1 p.m., and at 8 p.m. it was at the same point as at 7 p.m. Prove that the ship sails along a straight line.

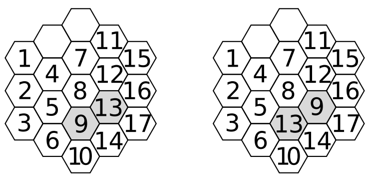
(A. Golovanov)

6. In the plane n segments with lengths a_1, a_2, …, a_n are drawn. Every ray beginning at the point O meets at least one of the segments. Let h_i be the distance from O to the i-th segment (not the line!) Prove the inequality

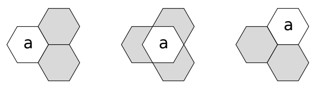
$$\frac{a_1}{h_1} + \frac{a_2}{h_2} + \ldots + \frac{a_n}{h_n} \geqslant 2\pi\,.$$

(F. Bakharev)

7. Hexagonal pieces numbered by positive integers are placed on the cells of a hexagonal board with side n. Two adjacent cells are left empty, and thanks to it some pieces can be moved. Two pieces with common sides exchanged places (see an example in the figure). Prove that if $n \geqslant 3$ the second arrangement cannot be obtained from the first one by moving pieces.



Note. Moving a piece a requires two adjacent empty cells. For instance, if they are on the right of a (left figure), a can be moved right till it touches an angle (middle figure), and then it can be moved upward right or downward right (right figure).



(R. Karpman, É. Roldán)

8. A positive integer n is given. Let A be the set of points x in the interval $(0, 1)$ such that $|x - \frac{p}{q}| > \frac{1}{n^3}$ for each rational fraction $\frac{p}{q}$ with denominator $q \leqslant n^2$. Prove that A is a union of intervals with the total length not exceeding $\frac{100}{n}$.

(F. Petrov)

Junior League

5. A graph contains p vertices numbered from 1 to p, and q edges numbered from $p+1$ to $p+q$. It turned out that for each edge the sum of the numbers of its ends and of the edge itself equals the same number s. It is also known that the numbers of edges starting in all vertices are equal. Prove that

$$s = \frac{1}{2}(4p + q + 3).$$

(*R. Figueroa-Centeno, R. Ichishima, F. Muntaner-Batle*)

6. An *Euclidean step* transforms a pair (a, b) of positive integers, $a > b$, to the pair (b, r), where r is the remainder when a is divided by b. Let us call the *complexity* of a pair (a, b) the number of Euclidean steps needed to transform it to a pair of the form $(s, 0)$. Prove that if $ad - bc = 1$, then the complexities of (a, b) and (c, d) differ at most by 2.

(*A. Golovanov*)

7. $3n$ people forming n families of a mother, a father and a child, stand in a circle. Every two neighbours can exchange places except the case when a parent exchanges places with his/her child (this is forbidden). For what n is it possible to obtain every arrangement of those people by such exchanges? The arrangements differing by a circular shift are considered distinct.

(*K. Kokhas*)

8. Circle ω lies inside the circle Ω and touches it internally at point P. Point S is taken on ω and the tangent to ω is drawn through it. This tangent meets Ω at points A and B. Let I be the centre of ω. Find the locus of circumcentres of triangles AIB.

(*P. Kozhevnikov, A. Zaslavsky*)

SOLUTIONS

Senior League

5. Let us draw (some) line through the positions of the ship at 1 p.m. at 7 p.m. The equation of this line has the form $Ax + By = C$ with at least one of the numbers A and B not being zero. The ship visits this line at least four times, that is, the polynomial

$$Af(t) + Bg(t) - C = 0$$

has at least four roots. But its degree is at most 3, therefore it is identically zero, and the ship stays on the line forever.

6. For the inequality to be sensible we assume that none of the segment contains O.

Note that the following *refinement rule* is valid. If any of the segments is partitioned into several parts, say, the first segment is partitioned into parts with lengths a_1' and a_2'' (where $a_1 = a_1' + a_2''$) and consider them as separate segments, the desired inequality will be strengthened. Indeed, $h_1' \geqslant h_1$: if the point of a_1 nearest to O belongs to a_1', then $h_1' = h_1$, otherwise $h_1' > h_1$. Similarly $h_2'' \geqslant h_1$ and thus $\frac{a_1}{h_1} = \frac{a_1'}{h_1} + \frac{a_1''}{h_1} \geqslant \frac{a_1'}{h_1'} + \frac{a_1''}{h_1''}$.

Let us draw rays from O through all the ends of the segments. These rays divide the plane into several angles and the segments into some parts so that each part of a segment lies in one angle entirely. We will consider those part as new segments. By the refinement rule it is enough to prove the inequality for the new (refined) segments. We will remove some segments so that each angle contains one segment, and show that the inequality is still valid.

L e m m a. Let $\alpha = \angle AOB$ be an angle in the triangle OAB, $OA \leqslant OB$, and the height from point O drops outside the interior of segment AB. Then $\frac{AB}{OA} \geqslant \alpha$.

P r o o f. The circular sector with radius OA and central angle α is entirely covered by the triangle AOB, therefore its area does not exceed the area of the triangle, i. e.

$$\frac{1}{2}OA^2 \cdot \alpha \leqslant S(AOB) \leqslant \frac{1}{2}OA \cdot AB.$$

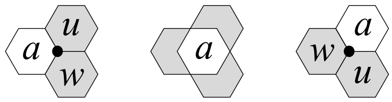
The desired inequality follows immediately.

C o r o l l a r y. Let $\alpha = \angle AOB$ be an angle in the triangle OAB, the altitude drawn from O has length h and its foot lies on the segment AB. Then $\frac{AB}{h} \geqslant \alpha$.

To prove the corollary it is enough to note that the altitude divides the triangle into two triangle, apply the lemma to these triangles and add the inequalities.

Back to our problem, we now apply the lemma to each of the sectors, add the inequalities and get the desired result.

7. Let us assume that one empty cell is covered by a transparent piece u, another empty cell is covered by a transparent piece v, and when a "real" piece a is moved, the three pieces are moved in cycle: a takes the place of u, u the place of v, and v the place of a. Thus one move performs a cyclic permutation of three cells (two of them transparent), which is an even permutation of all the cells, including empty ones.



Now we claim that if transparent pieces u and v return to their initial places, then each of them returns to its place (that is, they could not exchange places).

To distinguish the empty cells we draw the vector from the centre of u to the centre of v. Looking at the picture above we notice that a movement of real piece causes this vector to rotate by $\pm 120°$. For the transparent pieces to exchange their places, this vector should rotate by $180°$, which cannot be done by repeated rotations by $\pm 120°$.

Thus we established that if two real cells exchanged their places and all other real cells returned to their original positions, the transparent cells also returned to their positions, that is, we performed a transposition of two pieces on the set of all pieces, transparent included. This is, however, an odd permutation, which cannot be obtained by repeated even permutations.

8. Let us write in increasing order all irreducible fractions on $[0;1]$ with denominator not exceeding n^2. We make use of the following fact: if $\frac{a}{b} > \frac{c}{d}$ are two consecutive fractions in the obtained sequence, then $b + d \geqslant n^2$ and $ad - bc = 1$. To prove the first inequality it is enough to note that the fraction $\frac{a+c}{b+d}$ lying between $\frac{a}{b}$ and $\frac{c}{d}$ is not in our sequence. To prove the second one, we assume, without loss of generality, that $b > d$, and find $u < a$, $v < b$ such that $av - bu = 1$ (we can do that because $(a, b) = 1$). If $\frac{u}{v} \neq \frac{c}{d}$, it follows from $\frac{a}{b} > \frac{u}{v}$ that the difference $\frac{c}{d} - \frac{u}{v} = \frac{cv-du}{dv} \geqslant \frac{1}{dv}$ is less than $\frac{a}{b} - \frac{u}{v} = \frac{1}{bv}$, hence $d > b$, a contradiction. Thus the fraction $\frac{u}{v}$ is in fact the neighbour of $\frac{a}{b}$.

All the points of A belong to the intervals between neigbouring fractions with difference greater $\frac{2}{n^3}$ (we will call such intervals *long*). The set A is obtained by removing segments of length $\frac{1}{n^3}$ from both ends of each long interval. Let the ends of a long interval have denominators a and $b < a$. We know that $a + b > n^2$ and hence $a > \frac{n^2}{2}$, and the length of the interval is $\frac{1}{ab} > \frac{2}{n^3}$, hence $ab < \frac{n^3}{2}$ and $b < n$. For each denominator $b < n$ we have at most $2b$ long intervals ending in a fraction with denominator b. The denominator of the other end is greater than $\frac{n^2}{2}$, therefore, the length of such interval is less than $\frac{2}{bn^2}$, and the total length of such intervals is less than $\frac{4}{n^2}$. Multiplying this estimate by n, we see that the total length of all the long intervals – and, a fortiori, of all the intervals forming the set A – is less than $\frac{4}{n}$.

Junior League

5. Let r be the degree of (any) vertex. The numbers p, q, and r satisfy the relation

$$rp = 2q$$

(both sides equal the sum of degrees of all vertices). For each edge we find the sum of numbers of the ends of this edge and of the edge itself, and then add up all the sums we obtaim. The probles says that the sum total is qs. On the other hand, it equals the sum of numbers of all edges plus the sum of numbers of all vertices multiplied by r, i.e.

$$sq = (p+1)+\ldots+(p+q)+r(1+2+\ldots+p) = q\cdot\frac{2p+q+1}{2} + \frac{rp(p+1)}{2}.$$

Replacing rp by $2q$ and dividing by q we get the desired result.

6. Let $q = \left[\frac{c}{d}\right]$ be the quotient when c is divided by d. If $\left[\frac{a}{b}\right]$ also equals q, one Euclidean step transforms the pairs (a, b) and (c, d) to $(b, a - qb)$ and $(d, c - qd)$ also satisfying the condition (since $d(a - qb) - b(c - qd) = aq - bc = 1$), and it suffices to prove the claim for the new pairs.

Since an Euclidean step diminishes the sum of the numbers, eventually we either get a pair containing 0, or reach a situation where $\left[\frac{a}{b}\right] \neq \left[\frac{c}{d}\right]$.

Obviously the numbers in each of the pairs (a, b) and (c, d) are coprime if $ad - bc = 1$. Therefore, if one of the pairs contains 0, it is the pair $(1, 0)$. If the other pair also contains 0, the complexities of the pairs are equal. Otherwise, the other pair contains 1, and its complexity is greater by 1.

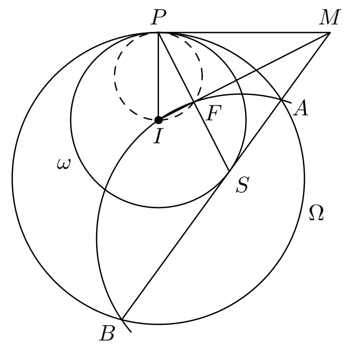
It remains to solve the problem for pairs (a, b) and (c, d) with $\left[\frac{a}{b}\right] > \left[\frac{c}{d}\right] = q$. We have $a \geqslant b(q+1)$ and $c < d(q+1)$, that is, $ad \geqslant bd(q+1) >$

$bc = ad-1$, hence $a = b(q+1)$. Since a and b are coprime, $b = 1$, $a = q+1$, $c = ad-1 = (a-1)d+(d-1)$. Therefore Euclidean step transforms (a, b) to $(1, 0)$, and (c, d) to $(d, d-1)$, which becomes $(1, 0)$ if $d = 2$, or $(d-1, 1)$ and then $(1, 0)$ if $d > 2$. In these cases the complexity of (c, d) is greater than that of (a, b) by 1 and 2 respectively.

7. There are no such n. In fact there are three classes of equivalent arangements. However, we will only find an invariant attaining three different values.

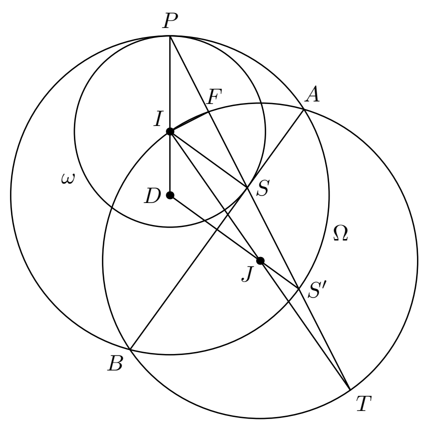
Let us think that the people stand in a row (but the leftmost and the rightmost persons are considered neighbours). Now we define the weight of a family to be 1, if the child stands to the left of both parents, 0, if s/he is to the right of them and 2 if s/he stands between them. It is easy to check that the total weight of all families modulo 3 does not change, thus, for instance, we cannot exchange places of a parent and a child preserving the position of other people.

8. Let D be the centre of Ω, R and r the radii of Ω and ω. Then the locus in question is the circle with centre D and radius $R - \frac{1}{2}r$ (possibly without one point corresponding to degenerate case).



L e m m a. Let F be the midpoint of PS. Then F belongs to the circumcircle of AIB.

P r o o f. Let M be the intersection of AB and the tangent to Ω at P. Then the triangle SPM is isosceles and the bisector MI contains P. Then the angle IFP is right and the circumcircle of IFP touches both Ω and ω at P. The circle Ω gives us $MA \cdot MB = MP^2$, and the circumcircle of IFP gives $MF \cdot MI = MP^2$. Comparing these equations we find that $MA \cdot MB = MF \cdot MI$, hence the result.



Let J be the centre of the circle AIB, and T the point on this circle diametrically opposite to I. Since the angle IFS is right, it is subtended by a diameter, i.e. the line PS passes through T. Let S' be the point where line PS meets Ω. In Ω we have the equality

$$AS \cdot SB = PS \cdot SS',$$

and in the circumcircle of AIB,

$$AS \cdot SB = FS \cdot ST.$$

Juxtaposing these equations and taking into account that $FS = \frac{1}{2}PS$, we see that $SS' = \frac{1}{2}ST$, therefore JS' is the midline of the triangle IST, $JS' = \frac{1}{2}IS = \frac{1}{2}r$, $JS' \parallel IS$. But the homothety with centre P which transforms ω to Ω, transforms line IS to line DS', therefore $DS' \parallel IS$. Thus J lies on DS'. Hence $DJ = DS' - JS' = R - \frac{1}{2}r$.