REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2023" (mathematics)

First day

Yakutsk 2023

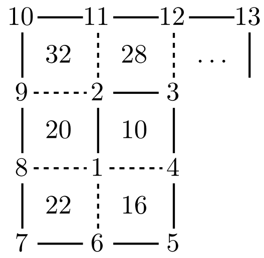
The booklet contains the problems of XXX International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical commission of Russian mathematical olympiad. The booklet was compiled by A. V. Antropov, S. L. Berlov, A. S. Golovanov, K. P. Kokhas. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

1. The numbers 1, 2, 3, ... are arranged in a spiral in the vertices of an infinite square grid (see figure). Then in the centre of each square the sum of the numbers in its vertices is placed. Prove that for each positive integer n the centres of the squares contain infinitely many multiples of n.



(*K. Kokhas*)

2. In a graph with n vertices every two vertices are connected by a unique path. For each two vertices u and v, let $d(u, v)$ denote the distance between u and v, i.e. the number of edges in the path connecting these two vertices, and $\deg u$ denote the degree of a vertex u. Let W be the sum of pairwise distances between the vertices, and D the sum of weighted pairwise distances:

$$D = \sum_{\{u,v\}} (\deg u + \deg v) d(u, v).$$

Prove that $D = 4W - n(n-1)$.

(*I. Gutman*)

3. Prove for integral $n \geqslant 2$ the inequality

$$\frac{\sqrt[3]{\frac{1}{n+1}} + \sqrt[3]{\frac{2}{n+1}} + \ldots + \sqrt[3]{\frac{n}{n+1}}}{n} \leqslant \frac{\sqrt[3]{\frac{1}{n}} + \sqrt[3]{\frac{2}{n}} + \ldots + \sqrt[3]{\frac{n-1}{n}}}{n-1}.$$

(*J. Liu*)

4. Two points A and B and line ℓ are fixed in the plane so that ℓ is not perpendicular to AB and does not intersect the segment AB. We consider all circles with a centre $O \notin \ell$ passing through A and B and meeting ℓ at some points C and D. Prove that all the circumcircles of triangles OCD touch a fixed circle.

(*S. Berlov*)

Junior League

1. Prove that for a, b, $c \in [0,1]$ the following inequality holds:

$$(1-a)(1+ab)(1+ac)(1-abc) \leqslant (1+a)(1-ab)(1-ac)(1+abc).$$

(G. Raposo)

2. Serge and Tanya want to show Masha a magic trick. Serge leaves the room. Masha writes down a sequence $(a_1, a_2, \ldots, a_n)$, where all a_k equal 0 or 1. After that Tanya writes down a sequence $(b_1, b_2, \ldots, b_n)$, where all b_k also equal 0 or 1. Then Masha either does nothing or says "Mutabor" and replaces both sequences: her own sequence by $(a_n, a_{n-1}, \ldots, a_1)$, and Tanya's sequence by $(1-b_n, 1-b_{n-1}, \ldots, 1-b_1)$. Masha's sequence is covered by a napkin, and Serge is invited to the room. Serge should look at Tanya's sequence and tell the sequence covered by the napkin. For what n Serge and Tanya can prepare and show such a trick? Serge does not have to determine whether the word "Mutabor" has been pronounced.

(A. Antropov, T. Gizatullin)

3. Point L inside triangle ABC is such that $CL = AB$ and $\angle BAC + \angle BLC = 180^\circ$. Point K on the side AC is such that $KL \parallel BC$. Prove that $AB = BK$.

(A. Antropov)

4. Two players play a game. They have $n > 2$ piles containing $n^{10}+1$ stones each. A move consists of removing all the piles but one and dividing the remaining pile into n nonempty piles. The player that cannot move loses. Who has a winning strategy, the player that moves first or his adversary?

(T. Abuku, K. Sakai, M. Shinoda, K. Suetsugu)

SOLUTIONS

Senior League

1. The squares of the grid form a spiral consisting of straight "corridors", which turn at right angles. It is easy to see that the numbers in the centres of adjacent squares differ by 4 and are divisible by 4. Therefore the squares of a corridor are filled with consecutive multiples of 4. It follows that any corridor long enough contains a multiple of n.

2. Asphalt plant lemma. Let T be an arbitrary tree with n vertices and α any of its vertices. Then

$$\sum_u (\deg u - 1)\ d(u, \alpha) = \sum_u d(u, \alpha) - (n-1).$$

P r o o f Imagine that the vertices are towns, edges are dirt roads, and we build an asphalt plant in α with the intention of surfacing all the roads. One edge requires one ton of asphalt, the surfacing itself is free but transportation is expensive, that is, carrying one ton of asphalt costs one dollar. Now, what will we spend on total asphaltization? The business plan is obvious: we carry the necessary asphalt to each city and surface all the roads going from it away from α. These expenses are counted in the left hand side of our equality. Now let us unring the bell: dismantle all the asphalt and carry it back to alpha. For each edge vu we remove one ton of asphalt from the road, place it in u (the far end of vu) and then carry it to the plant, thus spending $d(u, \alpha)$ dollars. We returned the pavement to factory settings, but the evacuation of each road cost 1 dollar more than its delivery. This is expressed by the right hand side. $\square$

Now we will deduce the claim from this lemma.

F i r s t s o l u t i o n. Let V — be the set of all vertices. Note that

$$\begin{aligned} D &= \sum_{\{u,v\}\subset V} (\deg u + \deg v) d(u,v) = \\ &= \sum_{v\in V}\sum_{u\in V} \deg u \cdot d(u,v) = \\ &= \sum_{v\in V}\sum_{u\in V} (\deg u - 1) d(u,v) + \sum_{v\in V}\sum_{u\in V} d(u,v) \underset{(*)}{=} \\ &= \sum_{v\in V}\Big(\sum_{u\in V} d(u,v) - (n-1)\Big) + 2W = \\ &= 4W - n(n-1). \end{aligned}$$

The lemma is used in the equality $(*)$.

S e c o n d s o l u t i o n. We will prove the equality $D = 4W - n(n-1)$ for every tree with n vertices by induction in n. The base $n = 1$ is trivial.

Now we prove the inductive step. Let a tree T' with n vertices be obtained from a tree T with $n-1$ vertices by joining a leaf vertex α to a vertex β. Then

$$\begin{aligned} D(T') &= \sum_{\{u,v\}\subset T'} (\deg_{T'} u + \deg_{T'} v) d(u,v) = \\ &= \sum_{\substack{\{u,v\}\subset T': \\ \alpha\notin\{u,v\}}} (\deg_{T'} u + \deg_{T'} v) d(u,v) + \sum_{\substack{u\in T': \\ u\neq\alpha}} (\deg_{T'} u + \deg_{T'} \alpha) d(u,\alpha) = \\ &= D(T) + \sum_{u\in T} d(u,\beta) + \sum_{\substack{u\in T': \\ u\neq\alpha}} (\deg_{T'} u + 1) d(u,\alpha), \end{aligned}$$

Here equalizing term $\sum_u d(u,\beta)$ comes from the relation $\deg_{T'}\beta = \deg_T \beta + 1$. Then

$$W(T') = \sum_{\{u,v\}\subset T'} d(u,v) = \sum_{\substack{\{u,v\}\subset T': \\ \alpha\notin\{u,v\}}} d(u,v) + \sum_{\substack{u\in T': \\ u\neq\alpha}} d(u,\alpha) = W(T) + \sum_u d(u,\alpha).$$

By the induction hypothesis we have

$$D(T) = 4W(T) - (n-1)(n-2).$$

The inductive step is proved as soon as we prove the equality

$$\sum_u (\deg u + 1) d(u,\alpha) + \sum_u d(u,\beta) = 4\sum_u d(u,\alpha) - 2(n-1). \qquad (*)$$

To simplify it, we note that

$$\sum_{u\in T} d(u,\beta) = \sum_{u\in T'} d(u,\alpha) - (n-1).$$

Removing three sums $\sum_u d(u,\alpha)$ from both sides of $(*)$, we obtain the asphalt plant lemma:

$$\sum_u (\deg u - 1)\ d(u,\alpha) = \sum_u d(u,\alpha) - (n-1).$$

3. Let $f(x) = \sqrt[3]{x}$. The function $f(x)$ is concave on $[0,1]$, that is,

$$(1-\alpha)f(x) + \alpha f(y) \leqslant f((1-\alpha)x + \alpha y) \qquad (*)$$

for $x, y, \alpha \in [0;1]$. You can notice that by looking on the graph of f on $[0;1]$, and prove it by differentiating f twice or cubing $(*)$ a little longer.

For each k from 1 to $n-1$ we apply (*) to $x = \frac{k}{n+1}$, $y = \frac{k+1}{n+1}$, $\alpha = \frac{k}{n}$. Since

$$\frac{n-1}{n}\cdot\frac{1}{n+1} + \frac{1}{n}\cdot\frac{2}{n+1} = \frac{1}{n},$$
$$\frac{n-2}{n}\cdot\frac{2}{n+1} + \frac{2}{n}\cdot\frac{3}{n+1} = \frac{2}{n}, \quad \text{and so on,}$$

we get the inequalities

$$\frac{n-1}{n}f\Big(\frac{1}{n+1}\Big) + \frac{1}{n}f\Big(\frac{2}{n+1}\Big) \leqslant f\Big(\frac{1}{n}\Big),$$
$$\frac{n-2}{n}f\Big(\frac{2}{n+1}\Big) + \frac{2}{n}f\Big(\frac{3}{n+1}\Big) \leqslant f\Big(\frac{2}{n}\Big), \quad \text{and so on.}$$

It remains to add up everything and divide by $n-1$.

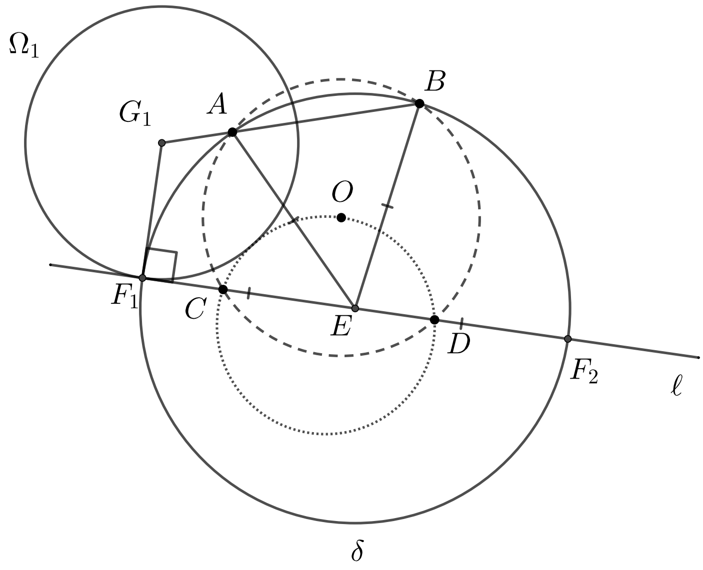
4. In fact, there are *two* such fixed circles. Let ω denote the variable circle passing through A and B.

We plan to construct circles Ω_1 and Ω_2 so that both are tangent to ℓ and preserved under inversion about any ω. Since ℓ is tangent to both our hypothetical circles Ω_1 and Ω_2, the image of ℓ under the inversion about ω, that is, the circumcircle of OCD, will also touch Ω_1 and Ω_2, thus solving the problem.

We will construct Ω_1 and Ω_2 now. Consider the point $E \in \ell$ such that $EA = EB$. Let the circle δ with centre E and radius EA meet ℓ at F_1 and F_2. Choosing F_1, we raise the perpendicular to ℓ at it, and let this perpendicular meet the line AB at G_1. The circle with centre G_1 and radius G_1F_1 is the desired Ω_1.

Indeed, the construction ensures that Ω_1 is tangent to ℓ. The power of G_1 with respect to this circle is $G_1F_1^2 = G_1A \cdot G_1B$, i. e., the inversion about Ω_1 transforms A and B into each other. This means that every circle containing A and B is preserved under inversion about Ω_1. Now, if ω is preserved under inversion about Ω_1, Ω_1 is preserved under inversion about ω (since both claims mean that ω and Ω_1 are perpendicular), q.e.d.

The circle Ω_2 is constructed similarly.



Junior League

1. F i r s t s o l u t i o n. For $a = 0$ the inequality is obvious. Let $a \neq 0$. The desired inequality is the product of two inequalities, "with pluses" and "with minuses":

$$(1+ab)(1+ac) \leqslant (1+a)(1+abc) \quad \text{and} \quad (1-a)(1-abc) \leqslant (1-ab)(1-ac).$$

Both are immediately checked by expanding, dividing by a and factorizing of the rest. For instance, let us expand the second inequality:

$$1 - a - abc + a^2bc \leqslant 1 - ab - ac + a^2bc.$$

Collecting all the terms on the right, we get $0 \leqslant 1 + bc - b - c$, i.e., $0 \leqslant (1-b)(1-c)$, which is clear.

S e c o n d s o l u t i o n. The problem can also be done by smoothing, that is, using the (well-known) fact that the sum of two positive real numbers whose product is fixed increases when their difference increases (this is merely the identity $(x+y)^2 = 4xy + (x-y)^2$). Let us expand partially:

$$(1-a)(1+ab+ac+a^2bc)(1-abc) \leqslant (1+a)(1-ab-ac+a^2bc)(1+abc)$$

and fix a. Now, if we replace b and c by 1 and bc, the outer factors on both sides do not change. It follows from the above observation that the middle factor on the left increases and that on the right decreases. To prove the inequality it suffices now to check it for $b = 1$. But with $b = 1$ the inequality becomes

$$(1-a)(1+a+ac+a^2c)(1-ac) \leqslant (1+a)(1-a-ac+a^2c)(1+ac)$$

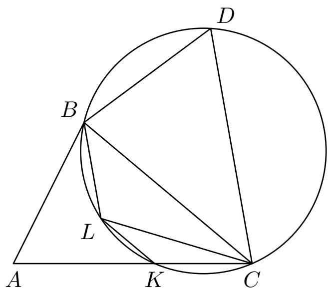
which is actually an identity.

2. A n s w e r: for even n.

Masha can write 2^n different sequences, some of them forming pairs connected by the operation "Mutabor", and some separate sequences that remain unchanged under this operation (of course, we will call them *palindromes*). Similarly, Tanya's 2^n sequences form some pairs of sequences transformed to each other by the operation "Mutabor", and separate sequences that remain unchanged under this operation (which we will call *antipalindromes*). To succeed with their trick, Tanya and Serezha have to assign to each Masha's sequence s some sequence $f(s)$ which will be Tanya's answer to s. Of course, distincts sequences $f(s)$ should be assigned to distinct s. Moreover, since Masha can say "Mutabor", two sequences forming a pair should be assigned to sequences forming a pair. It follows that antipalindromes should be assigned to palindromes.

When $n = 2k+1$ is odd, there are no antipalindromes (since b_{k+1} cannot be equal to $1 - b_{k+1}$), while some palindromes (for instance, sequences of n equal numbers) exist. In this case it is impossible to assign antipalindromes to palindromes, and the trick goes wrong.

If $n = 2k$ is even, we have 2^k palindromes and 2^k antipalindromes (since both are uniquely defined by the first k terms of a sequence), and the rest of the sequences, both Masha's and Tanya's, form $(2^n - 2^k)/2$ pairs. Thus Tanya and Serezha can simply list all the sequences, establish the desired one-to-one mapping and show the trick to Masha.



3. Let point D be symmetric to A with respect to line BC. Then $\angle BDC = \angle BAC$, therefore $\angle BDC + \angle BLC = 180°$, i. e. quadrilateral $BDCL$ is cyclic. Let $K' \neq C$ be the point where the circumcircle of $BDCL$ meets line AC. Note that $\angle LBC = \angle BCD = \angle BCA$ (the first equality follows from parallelism, the second one is true because A and D are symmetric). This equality implies that $LBCK'$ is an isosceles trapezoid, $LK' \parallel BC$, and therefore $K = K'$. Then $BK = BD$ since these chords subtend equal angles, and $BD = AB$ by symmetry.

4. A n s w e r: The first player wins.

Note that $n^{10} + 1 = n(n-1)\big(n^8 + n^7 + \ldots + n + 1\big) + n + 1$.

We claim that the winning positions in this game (that is, the positions one should left to the adversary to win) are those where all the numbers of stones in the piles leave remainders between 1 and $n-1$ when divided by $n(n-1)$. It follows immediately that the first player wins.

Indeed, the final position (where one cannot move) are those where each pile contains between 1 and $n-1$ stones, i.e. they have the form we described.

Now note that one cannot move from a position described above to another such position. In every such position the sum of the remainders left by the numers of stones in the piles when divided by $n(n-1)$ lies between n and $n(n-1)$, thus it cannot be obtained by dividing a pile in another such position.

Finally, we see that if a position is not of that type, at least in one of its piles the number of stones is congruent $\mathrm{mod}\, n(n-1)$ to some number between n and $n(n-1)$, and therefore can be presented as a sum of n numbers leaving upon division by $n(n-1)$ remainders between 1 and $n-1$.