REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2022" (mathematics)

Second day

Yakutsk 2022

The booklet contains the problems of XXIX International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by A. S. Golovanov, K. S. Ivanov, K. P. Kokhas, F. V. Petrov. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

5. Prove that a quadratic trinomial $x^2 + ax + b$ $(a, b \in \mathbb{R})$ cannot attain at ten consecutive integral points values equal to powers of 2 with non-negative integral exponent.

(*F. Petrov*)

6. Kostya marked the points $A(0,1)$, $B(1,0)$, $C(0,0)$ in the coordinate plane. On the legs of the triangle ABC he marked the points with coordinates $(\frac{1}{2},0)$, $(\frac{1}{3},0)$, $\ldots$, $(\frac{1}{n+1},0)$ and $(0,\frac{1}{2})$, $(0,\frac{1}{3})$, $\ldots$, $(0,\frac{1}{n+1})$. Then Kostya joined each pair of marked points with a segment. Sasha drew a $1 \times n$ rectangle and joined with a segment each pair of integer points on its border. As a result both the triangle and the rectangle are divided into polygons by the segments drawn. Who has the greater number of polygons: Sasha or Kostya?

(*M. Alekseyev*)

7. A $1 \times 5n$ rectangle is partitioned into *tiles*, each of the tile being either a separate 1×1 square or a *broken domino* consisting of two such squares separated by four squares (not belonging to the domino). Prove that the number of such partitions is a perfect fifth power.

(*K. Kokhas*)

8. In an acute triangle ABC the points C_m, A_m, B_m are the midpoints of AB, BC, CA respectively. Inside the triangle ABC a point P is chosen so that $\angle PCB = \angle B_mBC$ and $\angle PAB = \angle ABB_m$. A line passing through P and perpendicular to AC meets the median BB_m at E. Prove that E lies on the circumcircle of the triangle $A_mB_mC_m$.

(*K. Ivanov*)

Junior League

5. Each row of a 24×8 table contains some permutation of the numbers 1, 2, $\ldots$, 8. In each column the numbers are multiplied. What is the minimum possible sum of all the products?

(*C. Wu*)

6. The city of Neverreturn has N bus stops numbered 1, 2, $\ldots$, N. Each bus route is one-way and has only two stops, the beginning and the end. The route network is such that departing from any stop one cannot return to it using city buses.

When the mayor notices a route going from a stop with a greater number to a stop with a lesser number, he orders to exchange the number plates of its beginning and its end. Can the plate changing go on infinitely?

(*K. Ivanov*)

7. M is the midpoint of the side AB in an equilateral triangle ABC. The point D on the side BC is such that $BD : DC = 3 : 1$. On the line passing through C and parallel to MD there is a point T inside the triangle ABC such that $\angle CTA = 150°$. Find the angle MTD.

(*K. Ivanov*)

8. Eight poles stand along the road. A sparrow starts at the first pole and once in a minute flies to a neighbouring pole. Let $a(n)$ be the number of ways to reach the last pole in $2n + 1$ flights (we assume $a(m) = 0$ for $m < 3$). Prove that for all $n \geqslant 4$

$$a(n) - 7a(n-1) + 15a(n-2) - 10a(n-3) + a(n-4) = 0.$$

(*T. Amdeberhan, F. Petrov*)

SOLUTIONS

Senior League

5. It is easy to check that each monic quadratic trinomial $f(x)$ satisfies the relation

$$f(k+3) - f(k+2) - f(k+1) + f(k) = 4.$$

Suppose there exist such ten consecutive integers. Then at least five of them form a segment where the trinomial is monotonous. Let the maximum value of the trinomial in these four points is 2^n (n has to be at least 4). Supposing, without loss of generality, that the trinomial is increasing on the above segment and the maximim value $2^n = f(k+3)$, we obtain

$$4 = f(k+3) - f(k+2) - f(k+1) + f(k) \geqslant 2^n - 2^{n-1} - 2^{n-2} + 1 \geqslant 16 - 8 - 4 + 1 = 5,$$

a contradiction.

6. The reader will easily agree that replacing an $1 \times n$ rectangle by a $\sqrt{2} \times n$ rectangle will not change the answer (provided the sides of length n are still divided into n equal segments). To make matters worse, we position the rectangle vertically, more or less like the Monolith of "Space Odyssey" fame. Naturally this position requires three coordinates to describe, so the vertices of the rectangle will be at $(1,0,0)$, $(0,1,0)$, $(1,0,n)$, and $(0,1,n)$. The marked points will have the coordinates $(1,0,k)$ and $(0,1,k)$ for $0 \geq k \geq n$.

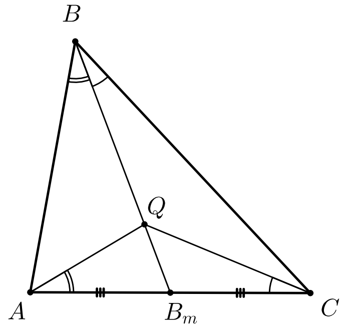
For better view we establish ourselves at $(0,0,-1)$ and project the marked points onto the plane $z = 0$. The images of points $(1,0,k)$ and $(0,1,k)$ under this projection will be $(\frac{1}{k+1},0,0)$ and $(0,\frac{1}{k+1},0)$. In other words, what we see in the plane $z = 0$ is the original triangle ABC with its marked points except one small triangle in the corner.

Thus our projection provides a bijection between all the parts of the rectangle and all but one parts of the triangle. This, of course, means that the number of parts in the triangle was greater by 1.

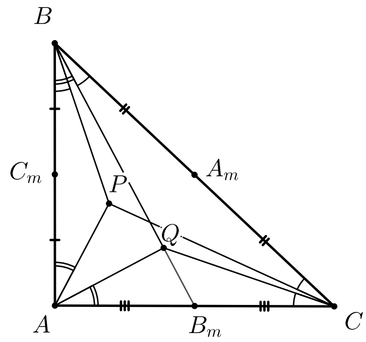
7. For each $r = 1,2,3,4,5$ let us consider the unit squares on the positions r, $r+5$, $r+10$, $\ldots$, $r+5(n-1)$. Forming five $1 \times n$ rectangles of these squares we see that each rectangle is partitioned into original tiles (with the only difference that we have unbroken the broken dominoes). Conversely, from any five partitions of $1 \times n$ rectangles into unit squares and dominoes the original partition can be reconstructed. Thus the number in question is the fifth degree of the number of ways to partition a $1 \times n$ rectangle into unit squares and dominoes.

8. Since the triangle ABC is acute, its median BB_m is greater than AB_m. This enables us to mark on the segment BB_m the point Q such that

$$\frac{1}{2}AC = QB_m = \frac{AB_m^2}{BB_m} = \frac{AB_m}{BB_m} \cdot AB_m < 1 \cdot AB_m < BB_m.$$



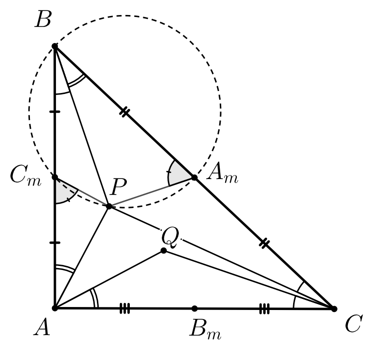
The equality $QB_m = \dfrac{AB_m^2}{BB_m}$ is equivalent to $\dfrac{QB_m}{AB_m} = \dfrac{AB_m}{BB_m}$ and thus implies the similarity of the triangles QAB_m and ABB_m. In particular,



$$\angle QAB_m = \angle ABB_m. \qquad (*)$$

Similarly $\angle QCB_m = \angle CBB_m$. (The reader can try to prove these relations finding the power of a carefully chosen point.)

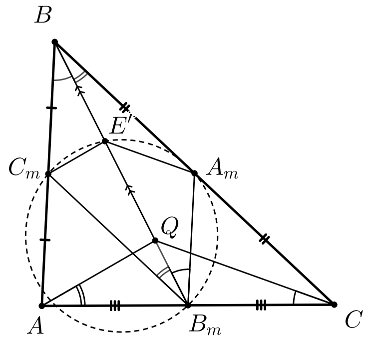
Let $\angle CBB_m = \alpha$, $\angle ABB_m = \beta$. Then $\angle PAB = \beta$ and (in view of $(*)$) $\angle QAC = \beta$. Similarly, $\angle PCB = \alpha$, $\angle QCA = \alpha$. It means that Q and P are isogonal conjugates, $\angle ABP = \alpha$, $\angle PBC = \beta$.



The triangles ABP and BCP have equal angles $\angle PAB = \angle PBC = \beta$ and $\angle PBA = \angle PCB = \alpha$. therefore they are similar and $\angle AC_mP = \angle BA_mP$ because of this similarity. It follows that the quadrilateralBA_mPC_m is cyclic.

Let E' be the midpoint of BQ. Then $E'C_m$ and $E'A_m$ are midlines in the triangles ABQ и CBQ, $E'C_m \parallel AP$ and consequently

$$\angle C_mE'A_m = \angle AQC = 180^\circ - \alpha - \beta.$$

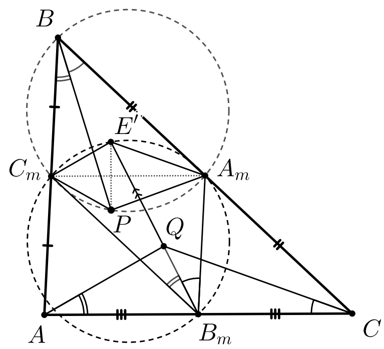


Since $BA_mB_mC_m$ is a parallelogram, $\angle A_mB_mC_m = \angle A_mBC_m = \alpha + \beta$ и $BB_mC_m = \beta$, $BB_mA_m = \alpha$.

We see that

$$\angle C_mE'A_m + \angle C_mB_mA_m = \\ = 180^\circ - \alpha - \beta + \alpha + \beta = 180^\circ,$$

which means that $E'A_mB_mC_m$ is cyclic.



The triangles BC_mA_m and $B_mA_mC_m$ are equal, and so are their circumcircles. In these equal circles the chords $E'C_m$ and PC_m subtend angles equal to β, while the chords $E'A_m$ and PA_m subtend angles equal to α, therefore $E'C_m = PC_m$ and $E'A_m = PA_m$. Then C_mA_m is the perpendicular bisector of PE', that is, $E'P \perp C_mA_m \parallel AC$, therefore E' coincides with the original point E, q.e.d.

Junior League

5. Ответ: $8 \cdot (8!)^3$. Let $P_1, \ldots, P_8$ be the products of numbers in the columns. It follows from AM–GM inequality that

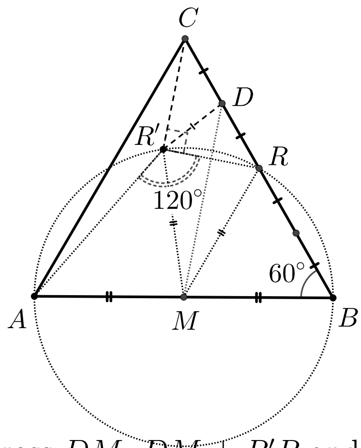
$$P_1 + P_2 + \ldots + P_8 \underset{(1)}{\geqslant} 8\sqrt[8]{P_1P_2\ldots P_8} \underset{(2)}{=} 8\sqrt[8]{(8!)^{24}} = 8 \cdot (8!)^3.$$

Here the equality (2) holds because $P_1P_2\ldots P_8$ is the product of all the numbers in the table. The only case of equality in the AM–GM inequality is when all the variables are equal. Therefore (1) becomes an equality if $P_1 = P_2 = \ldots = P_8$, and since this case is easily attained by putting in the table thrice any permutation and all its cyclic shifts, this very case gives the minimum sum of the products.

6. We may think that when plates are exchanged, the plate with the greater number is transported to its new destination by bus. It follows that the plate with number N will be exchanged a finite number of times. Indeed, if it makes infinitely many moves, at some moment it must visit a stop it has already visited, a contradiction.

When the plate number N ceases moving we can apply the same argument to the plate number $N-1$ and so on, down to the plate number 1. Thus at some moment all the plates will stop their movement, ending the whole process.

7. Ответ: $120°$.

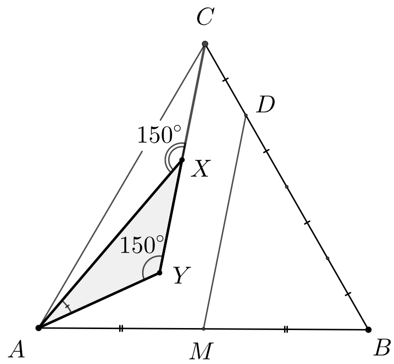


Let R be the midpoint of BC. Instead of T we consider the point R' symmetrical to R across DM. We will prove that $CR' \parallel DM$ and $\angle AR'C = 150°$, and finally that there is only one point in the triangle satisfying these conditions. It will follow that $R' = T$.

0. Finding the answer. The equations $RB = MB = AB/2$ and $\angle RBM = \angle CBA = 60°$ mean that the triangle RBM — равносторонний. Therefore $\angle MRB = 60°$ and $\angle DRM = 180° - \angle MRB = 120°$. Because of symmetry the desired angle $\angle DR'M = \angle DRM = 120°$.

1. Proving $CR' \parallel DM$. Since R and R' are symmetric across DM, $DM \perp R'R$ and $R'D = RD$. This means that the median $R'D$ in the triangle CRR' is equal to half the side CR, thus $\angle CR'R = 90°$, i. e. $CR' \perp R'R \perp DM$.

2. Proving $\angle AR'C = 150°$. By symmetry $MR' = MR = AB/2 = AM$, i. e. A, R', R, B lie on the circle with diameter AB, hence $\angle AR'R = 180° - \angle ABR = 180° - 60° = 120°$. As we already noted, $\angle CR'R = 90°$. Therefore the equality $\angle AR'R + \angle RR'C + \angle CR'A = 360°$ implies $\angle AR'C = 360° - 120° - 90° = 150°$.



3. Proving $R' = T$. Suppose the contrary. Then the line passing through C and parallel to DM contains two different points R' and T such that $\angle ATC = \angle AR'C = 150°$. Let X be one of these points that is nearer C, and Y the other one. The angle AXC is exterior for the triangle AXY, therefore

$$150° = \angle AXC = \angle AYX + \angle XAY > \angle AYX = \angle AYC = 150°,$$

a contradiction.

8. We introduce a little more detailed notation. Let $b_k(n)$ be the number of ways to reach the k-th pole in n flights. Then obviously $b_k(n) = 0$ if k and n have the same parity, and

$$\begin{aligned} b_k(n) &= b_{k-1}(n-1) + b_{k+1}(n-1) \quad \text{for } 1 < k < 8, \\ b_1(n) &= b_2(n-1), \\ b_8(n) &= b_7(n-1). \end{aligned} \qquad (*)$$

Clearly $a(n) = b_8(2n+1)$. Then

$$\begin{aligned}a(n) - 7a(n-1) &= b_8(2n+1) - 7b_8(2n-1) = \\ &= b_7(2n) - 7b_8(2n-1) = \\ &= \big(b_6(2n-1) + b_8(2n-1)\big) - 7b_8(2n-1) = \\ &= b_6(2n-1) - 6b_8(2n-1) = \\ &= b_5(2n-2) - 5b_7(2n-2).\end{aligned}$$

Let us extend our attention to $a(n-2)$ and further reduce the number of flights by unscrupulous use of $(*)$:

$$\begin{aligned}a(n) - 7a(n-1) + 15a(n-2) &= b_5(2n-2) - 5b_7(2n-2) + 15b_8(2n-3) = \\ &= b_4(2n-3) - 4b_6(2n-3) + 10b_8(2n-3) = \\ &= b_3(2n-4) - 3b_5(2n-4) + 6b_8(2n-4).\end{aligned}$$

And on, and on in the same vein:

$$\begin{aligned}&a(n) - 7a(n-1) + 15a(n-2) - 10a(n-3) = \\ &\qquad = b_3(2n-4) - 3b_5(2n-4) + 6b_8(2n-4) - 10b_8(2n-5) = \\ &\qquad = b_2(2n-5) - 2b_4(2n-5) + 3b_6(2n-5) - 4b_8(2n-5) = \\ &\qquad = b_1(2n-6) - b_3(2n-6) + b_5(2n-6) - b_7(2n-6).\end{aligned}$$

Turning finally to the case of $2n-7$ flights, we get

$$\begin{aligned}&a(n) - 7a(n-1) + 15a(n-2) - 10a(n-3) + a(n-4) = \\ &\qquad = b_1(2n-6) - b_3(2n-6) + b_5(2n-6) - b_7(2n-6) - b_8(2n-7) = 0.\end{aligned}$$