REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2022" (mathematics)

First day

Yakutsk 2022

The booklet contains the problems of XXIX International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical commission of Russian mathematical olympiad. The booklet was compiled by A. S. Golovanov, K. S. Ivanov, A. I. Khabrov, K. P. Kokhas, A. S. Kuznetsov. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

1. Some of 100 towns of a kingdom are connected by roads. It is known that for each two towns A and B connected by a road there is a town C which is not connected by a road with at least one of the towns A and B. Determine the maximum possible number of roads in the kingdom.

(*P. Qiaoa, X. Zhan*)

2. Two circles ω_1 and ω_2 of different radii touch externally at L. A line touches ω_1 at A and ω_2 at B (the points A and B are different from L). A point X is chosen in the plane. Y and Z are the second points of intersection of the lines XA and XB with ω_1 and ω_2 respectively. Prove that all X such that $AB \parallel YZ$ belong to one circle.

(*K. Ivanov*)

3. Is there a colouring of all positive integers in three colours so that for each positive integer the numbers of its divisors of any two colours differ at most by 2?

(*A. Golovanov*)

4. For every positive $a_1, a_2, \ldots, a_6$ prove the inequality

$$\sqrt[4]{\frac{a_1}{a_2+a_3+a_4}} + \sqrt[4]{\frac{a_2}{a_3+a_4+a_5}} + \ldots + \sqrt[4]{\frac{a_6}{a_1+a_2+a_3}} \geqslant 2.$$

(*A. Khrabrov*)

Junior League

1. Arnim and Brentano have a little vase with 1500 candies on the table and a huge sack with spare candies under the table. They play a game taking turns, Arnim begins. At each move a player can either eat 7 candies or take 6 candies from under the table and add them to the vase. A player cannot go under the table in two consecutive moves. A player is declared the winner if he leaves the vase empty. In any other case, if a player cannot make a move in his turn, the game is declared a tie. Is there a winning strategy for one of the players?

(*A. Golovanov*)

2. Given are integers a, b, c and an odd prime p. Prove that p divides $x^2 + y^2 + ax + by + c$ for some integers x and y.

(*A. Golovanov*)

3. Bisectors of a right triangle ABC with right angle B meet at point I. The perpendicular to IC drawn from B meets the line IA at D; the perpendicular to IA drawn from B meets the line IC at E. Prove that the circumcentre of the triangle IDE lies on the line AC.

(*A. Kuznetsov*)

4. Several *good* points, several *bad* points and several segments are drawn in the plane. Each segment connects a good point and a bad one; at most 100 segments begin at each point. We have paint of 200 colours. One half of each segment is painted with one of these colours, and the other half with another one. Is it always possible to do it so that every two segments with common end are painted with four different colours?

(*M. Qi, X. Zhang*)

SOLUTIONS

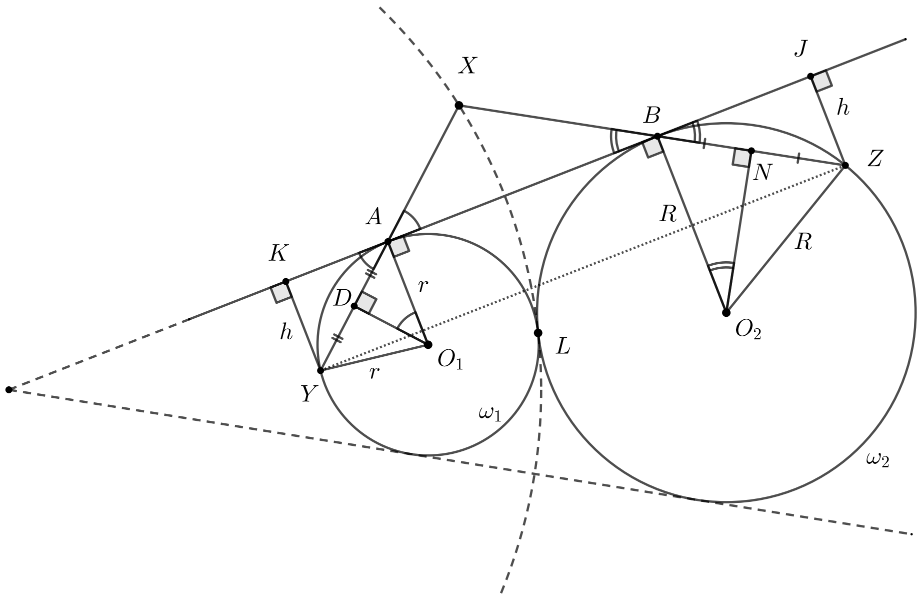
Senior League

1. A n s w e r: $98 \cdot 100/2$ roads. This number of roads is obtained if the towns form 50 pairs and two towns are connected by a road if and only if they do not form a pair: thus we get $99 \cdot 100/2 - 50 = 98 \cdot 100/2$ roads.

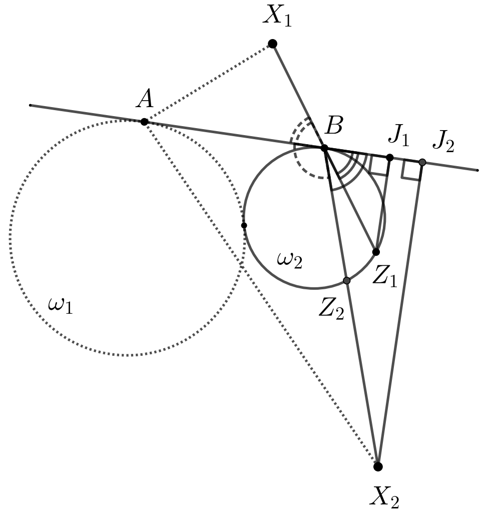
There cannot be two towns with 99 roads going from each (if they are A and B we immediately have a contradiction). If there is a town connected to all the others, the total number of roads starting in all the towns does not exceed $99 + 99 \cdot 98$. However, in this number each road is counted twice, therefore it must be even and thus not greater than $98 + 99 \cdot 98 = 98 \cdot 100$, while the number of roads does not exceed half that number.

Finally, if there is no town with 99 roads, the number of roads in each town does not exceed 98, and the number of all roads does not exceed $98 \cdot 100/2$.

2. Let O_1 and O_2 be the centres of ω_1 and ω_2, r and R their respective radii. Since $AB \parallel YZ$, the points Z and Y lie at the same distance from AB. We denote this distance by h.



Let K, D, N, J be the feet of perpendiculars dropped from Y, O_1, O_2, J to the lines AB, AY, BZ, AB, respectively. Obviously N is the midpoint of BZ, and $JZ = h$. If $\angle YAK = \alpha$, $\angle ZBJ = \beta$, then $\angle ZBO_2 = 90° - \beta$, $\angle BO_2N = 90° - (90° - \beta) = \beta$. Now we see that $\dfrac{JZ}{BZ} = \dfrac{h}{BZ} = \sin\beta$, $\dfrac{BN}{BO_2} = \dfrac{BZ/2}{R} = \sin\beta$, hence $\sin^2\beta = \dfrac{h}{2R}$. There are different positions of X, and the angle XBA is not always equal to β (the figure shows two possible positions of X, denoted by X_1 and X_2. It may be seen that $\angle X_1BA = \beta$ and $\angle X_2BA \neq \beta$).



This difficulty can be circumvented: both $\angle XBA$ and $\angle JBZ = \beta$ are formed by the lines XB and XA, therefore $\angle XBA$ can be equal either β or $180° - \beta$. In any case $\sin\angle XBA = \sin\beta$. Thus we have $\sin^2\angle XBA = \dfrac{h}{2R}$. Similarly, $\sin^2\angle XAB = \dfrac{h}{2r}$. Applying the law of sines to the triangle XAB, we see that $\dfrac{XA}{XB} = \dfrac{\sin\angle XBA}{\sin\angle XAB} = \sqrt{\dfrac{r}{R}}$, independently of X.

It is well known that if two points A, B and a positive $d \neq 1$ are fixed, the locus of points X such that $AX : BX = d$ is a circle called *circle of Apollonius*. Thus the point X in question lies on the circle of Apollonius for the points A and B and the coefficient $d = \sqrt{\frac{r}{R}} \neq 1$.

Note: The centre of this circle is the intersection point of the common external tangents to ω_1 and ω_2.

3. A n s w e r: yes. It is even possible to make the numbers in question differ at most by 1.

Let us colour a positive integer with colour r ($r = 0, 1, 2$) if the number of prime factors in the prime factorization of n leaves the remainder r upon division by 3. In other words,

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

is coloured with the colour $\alpha_1 + \alpha_2 + \dots + \alpha_k \bmod 3$.

We will check now that this colouring is valid.

F i r s t s o l u t i o n. Let $a_{n,0}$, $a_{n,1}$ и $a_{n,2}$ be the numbers of divisors of n such that their number of prime factors leaves the remainder 0, 1, and 2 respectively upon division by 3 (i. e. the numbers of divisors of colours 0, 1, and 2). We will prove that the difference of any two of these numbers does not exceed 1. The proof is by induction on the number of *different* prime divisors of n. If the number of prime divisors is 0, then $n = 1$, $a_{n,0} = 1$, $a_{n,1} = 0$, $a_{n,2} = 0$. Suppose the statement is true for some n, and let $m = np^k$, where p is a prime not dividing n.

All divisors of m form $k + 1$ groups $G_0, \dots, G_k$ so that for each s, $0 \leq s \leq k$, the group G_s is contains the divisors divisible by p^s and not divisible by p^{s+1}. If $s \equiv 0 \pmod 3$, the groups G_s contains $a_{n,0}$ divisors of colour 0, $a_{n,1}$ divisors of colour 1 and $a_{n,2}$ of colour 2; if $s \equiv 1 \pmod 3$, then G_s contains $a_{n,2}$, $a_{n,0}$, $a_{n,1}$ divisors of colours 0, 1, 2 respectively; if $s \equiv 2 \pmod 3$, $a_{n,1}$, $a_{n,2}$ и $a_{n,0}$ divisors of these colours. Clearly a union of the form $G_k \cup G_{k+1} \cup G_{k+2}$ contains equal number of divisors of all three colours. If $s \equiv 2 \pmod 3$, all the groups can be split into such triplets, and m has equal number of divisors of all three colours. If $s \equiv 0 \pmod 3$, only one group, namely G_s, remains outside such triplets, and the differences between numbers of divisors of the three colours are equal to those of $a_{n,0}$, $a_{n,1}$, $a_{n,2}$. Lastly, if $s \equiv 1 \pmod 3$, the two group outside the triplets are G_{s-1} and G_s; they contain $a_{n,1} + a_{n,2}$ divisors of colour 0, $a_{n,2} + a_{n,0}$ divisors of colour 1 and $a_{n,0} + a_{n,1}$ divisors of colour 2. The differences of these numbers also equal to those of $a_{n,0}$, $a_{n,1}$, $a_{n,2}$, that is, do not exceed 1 as desired.

Second solution. Let ε be a (not necessarily real) cubic root of 1. If a positive integer n is a product of k primes we define $f(k) = \varepsilon^k$. The function f is multiplicative (that is, $f(ab) = f(a)f(b)$ when a and b are coprime). It is known then that its *summatory function* $F(n) = \sum\limits_{d|n} f(d)$ is also multiplicative. This function by definition equals

$$F(n) = a_0 + a_1\varepsilon + a_2\varepsilon^2, \qquad (*)$$

where a_r is the number of divisors of n of colour r. On the other hand, $F(n)$ equals the product of the numbers of the form $F(p^s)$, where p^s is the highest power of a prime p dividing n. If $\varepsilon \neq 1$, the number $F(p^s) = 1 + \varepsilon + \ldots + \varepsilon^s$ does not exceed 1 in absolute value (in fact, $F(p^s)$ always equals 0, 1, or $1 + \varepsilon = -\varepsilon^2$). Substituting for ε in $(*)$ all cubic roots of 1, i. e, 1, $\rho = \frac{-1+\sqrt{3}i}{2}$ и ρ^2, we see that $a_0 + a_1 + a_2 = d(n)$, $|a_0 + a_1\rho + a_2\rho^2| \leqslant 1$, $|a_0 + a_1\rho^2 + a_2\rho| \leqslant 1$. Multiplying these formulas by cubic roots of unity and adding we get the desired result.

4. Lemma. $\sqrt[4]{\dfrac{x^2}{y^2+z^2+t^2}} \geqslant \dfrac{2x}{x+y+z+t}$ for positive x, y, z, t.

Proof

$$\frac{x+y+x+t}{2x} = \frac{1}{2}\left(1 + \frac{y+z+t}{x}\right) \geqslant \sqrt{\frac{y+z+t}{x}} = \sqrt[4]{\frac{(y+z+t)^2}{x^2}} \geqslant \sqrt[4]{\frac{y^2+z^2+t^2}{x^2}}.$$

Inverting both sides we get the desired result.

Let $x_1 = \sqrt{a_1}$, $x_2 = \sqrt{a_2}$ and so on, $S = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$. Applying the lemma to each term of the sum and adding the resulting inequalities, we get

$$\sqrt[4]{\frac{x_1^2}{x_2^2+x_3^2+x_4^2}} + \sqrt[4]{\frac{x_2^2}{x_3^2+x_4^2+x_5^2}} + \ldots + \sqrt[4]{\frac{x_6^2}{x_1^2+x_2^2+x_3^2}} \geqslant$$
$$\geqslant \left(\frac{2x_1}{x_1+x_2+x_3+x_4} + \frac{2x_2}{x_3+x_4+x_5+x_6} + \ldots + \frac{2x_6}{x_6+x_1+x_2+x_3}\right) \geqslant$$
$$\geqslant 2\left(\frac{x_1}{S} + \frac{x_2}{S} + \ldots + \frac{x_6}{S}\right) = 2,$$

as desired.

Junior League

1. Brentano has a winning strategy.

Suppose the vase contains $15k$ candies before Arnim's move (k is a positive integer). We will see that Brentano can leave $15(k-1)$ candies after two moves of each player, and the vase does not become empty in the process.

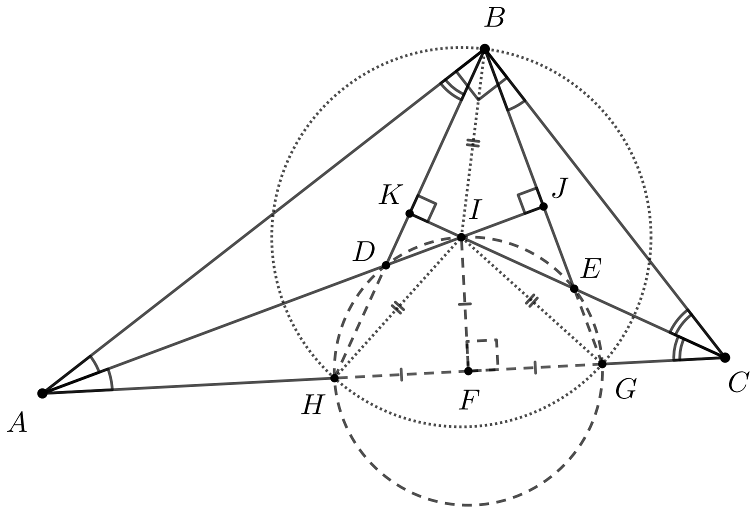
If Arnim adds 6 candies by his first move, next times he has to eat 7. Then Brentano should eat 7 candies twice, thus leaving $15k - 15$ candies. If Arnim eats 7 candies by his first move, Brentano also eats 7, leaving $15k - 14 > 0$ candies. If after that Arnim can eat 7 candies more (leaving $15k - 21 \neq 0$ candies in the vase), Brentano adds 6 and gets $15k - 15$. Otherwise Arnim adds 6, then Brentano eats 7 and gets the same result.

Repeating this 100 times Brentano wins.

2. Evaluating $f(x) = x^2 + ax + c$ for $x = 0, 1, \dots, p-1$ we get at least $\frac{p+1}{2}$ different remainders upon division by p. Indeed, if x_1 and x_2 are two distinct integers between 0 and $p-1$, and p divides $f(x_1) - f(x_2) = x_1^2 + ax_1 + c - (x_2^2 + ax_2 + c) = (x_1 - x_2)(x_1 + x_2 + a)$, then p also divides $x_1 + x_2 + a$, that is, for each x_1 between 0 and $p-1$ there is at most one x_2 on the same segment such that $f(x_2)$ and $f(x_1)$ leave the same remainder upon division by p.

Similar argument shows that the values of the polynomial $g(y) = -y^2 - by$ leave at least $\frac{p+1}{2}$ different remainders upon division by p. We have constructed two sets of remainders upon division by p, each one containing more than half of all remainders; these sets must have a common element. Therefore p divides $f(x) - g(y)$ for some integers x and y, q.e.d.

3. Let K and J be te feet of perpendiculars dropped from B to the lines IC и IA respectively, H and G the respective points where these perpendiculars meet AC. We denote by F the midpoint of HG.



Let $\angle GAI = \angle IAB = \alpha$ and $\angle BCI = \angle ICA = \beta$. The sum of angles in the triangle ABC is $90^\circ + 2\alpha + 2\beta = 180^\circ$, hence $\alpha + \beta = 45^\circ$. Applying the same argument to the triangle ABJ we get $\angle ABJ = 90^\circ - \alpha$ and therefore $\angle GBC = \angle ABC - \angle ABJ = 90^\circ - (90^\circ - \alpha) = \alpha$. Similarly $\angle ABH = \beta$. Thus $\angle HBG = \angle ABC - (\angle ABH + \angle GBC) = 90^\circ - 45^\circ = 45^\circ$.

Clearly AJ is both a bisector and an altitude in the triangle ABG. It follows that it is also the perpendicular bisector of BG. Similarly, KC is the perpendicular bisector of BH. Thus I is the common point of perpendicular bisectors to the sides of the triangle HBG. Then the line IF is a perpendicular bisector in the triangle HIG, and therefore its median and altitude, so the triangle HIG is isosceles.

Since I is the circumcentre of HBG, $\angle HIG = 2\angle HBG = 90^\circ$, i. e. HIG is not only isosceles but also right, and its angles are $45^\circ, 45^\circ, 90^\circ$. IF is the median of a right triangle, therefore $HF = IF = GF$, i. e. H, I, G belong to a circle with centre F and radius FH. To see that D and E belong to this circle, we look at the angles of the triangle BJD, where we have $\angle BDJ = 180^\circ - 90^\circ - 45^\circ = 45^\circ$, that is, $\angle HDI = 180^\circ - 45^\circ = 135^\circ$. Angle $\angle IGH = 45^\circ$ is already found. Thus we see that $\angle HDI + \angle HGI = 180^\circ$, that is, the quadrilateral $IGHD$ is cyclic. In the same way it is proved that E belongs to our circle.

Note. There are other solutions, partly coinciding with this one. For instance, one can note that the quadrilateral $AHIB$ is cyclic, or that the common point of DG and AB belongs to the circumcircle of the triangle HBG.

4. Answer: yes.

We define the bipartite graph T so that its vertices and edges correspond to the (good and bad) points and segments in the problem. We denote its parts by V_1 and V_2. We need also the graph T', a copy of T. For each vertex A of T the corresponding vertex of T' is denoted by A', the parts of T' are V_1' and V_2'. The union of T and T' is a bipartite graph; we will consider the union of V_1 and V_2' as one part and the union of V_2 and V_1' as another part of this graph. If a vertex A in T has degree $d < 100$, the vertex A' lies in the other part and has the same degree. We can add $100 - d$ edges between A and A' and repeat this procedure until we get a bipartite graph Q such that T is its subgraph and all the vertices of Q have degree 100.

The graph Q is *regular* (that is, all its vertices have the same degree). Thus we have constructed a *regular* graph. This graph can have multiple edges; the reader is advised to check that our argument is not affected by it.

We need the following well-known *Hall's theorem.* If a bipartite graph G satisfies Hall's condition, then G contains a set R of edges such that each vertex of G is the end of exactly one edge in R. Such a set is called *a perfect matching* in G. The Hall's condition requires that for each k and each k vertices belonging to the same part of G, the number of vertices adjacent to them is at least k.

Let us check the Hall's condition for the graph Q. If for k vertices in one part there are only $\ell < k$ vertices adjacent to them, the edges between these vertices have exactly $100k$ ends in one part and at most $100\ell < 100k$ vertices in another part, a contradiction.

Thus Q admits a perfect matching. We can colour all the edges of this matching with the first colour and remove them from Q. A regular graph remains with vertices of degree 99. In this graph we can again find a perfect matching, colour its edges with the second colour and remove them. Repeating this operation 100 times (we like simple monotonous work) we colour all the edges of Q with 100 colours so that all the edges beginning in each vertex have different colours. Such colouring is called *edge colouring* of Q.

Since T is a subgraph of Q, it also admits an edge colouring with 100 colours.

It remains to split each colour s into two hues s_1 and s_2 and colour two halves of each edge of colour s with different hues s_1 and s_2.