REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2020" (mathematics)

Second day

Yakutsk 2020

The booklet contains the problems of XXII International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by S. L. Berlov, A. S. Golovanov, S. V. Ivanov, K. P. Kokhas, A. S. Kuznetsov, N. Y. Vlasova. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

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Senior league

5. Coordinate axes (without any marks, with the same scale) and the graph of a quadratic trinomial $y = x^2 + ax + b$ are drawn in the plane. The numbers a and b are not known. How to draw a unit segment using only ruler and compass?

(*S. Berlov*)

6. An isosceles triangle ABC ($AB = BC$) is given. Circles ω_1 and ω_2 with centres O_1 and O_2 lie in the angle ABC and touch the sides AB and CB at A and C respectively, and touch each other externally at point X. The side AC meets the circles again at points Y and Z. O is the circumcentre of the triangle XYZ. Lines O_2O and O_1O intersect lines AB and BC at points C_1 and A_1 respectively. Prove that B is the circumcentre of the triangle A_1OC_1.

(*E. Lopatin*)

7. Several policemen try to catch a thief who has $2m$ accomplices. To that end they place the accomplices under surveillance. In the beginning, none of the accomplices are followed by the police. Every morning each policeman places under his surveillance one of the accomplices. Every evening the thief stops trusting one of his accomplices. The thief is caught if by the m-th evening some policeman shadows exactly those m accomplices who are still trusted by the thief. Prove that to guarantee the capture of the thief at least 2^m policemen are needed.

(*W. B. Kinnersley, D. B. West*)

8. The degrees of polynomials P and Q with real coefficients do not exceed n. These polynomial satisfy the identity

$$P(x)x^{n+1} + Q(x)(x+1)^{n+1} = 1.$$

Determine all possible values of $Q(-\frac{1}{2})$.

(*K. Dilcher, M. Ulas*)

Junior League

5. See problem 5 of senior league.

6. AK and BL are altitudes of an acute triangle ABC. Point P is chosen on the segment AK so that $LK = LP$. The parallel to BC through P meets the parallel to PL through B at point Q. Prove that $\angle AQB = \angle ACB$.

(*S. Berlov*)

7. How many positive integers N in the segment $[10, 10^{20}]$ are such that if all their digits are increased by 1 and then multiplied, the result is $N + 1$?

(*F. Bakharev*)

8. In a horizontal strip $1 \times n$ made of n unit squares the vertices of all squares are marked. The strip is partitioned into parts by segments connecting marked points and not lying on the sides of the strip. The segments can not have common inner points; the upper end of each segment must be either above the lower end or further to the right. Prove that the number of all partitions is divisible by 2^n. (The partition where no segments are drawn, is counted too.)

(*E. Robeva, M. Sun*)

SOLUTIONS

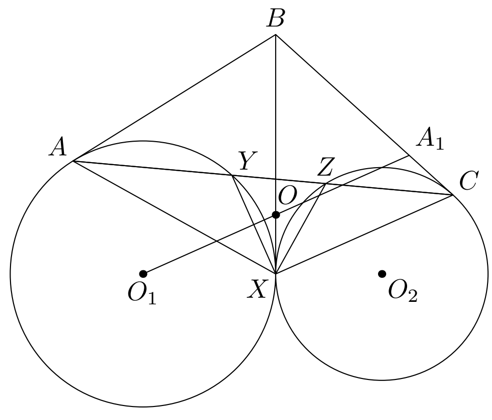
Senior League

5. First solution. Let us draw a line parallel to Ox and intersecting the graph. Let P and Q be the intersection points, then the perpendicular bisector of PQ meets the parabola at its vertex V. Now we shall draw a line through V parallel to the line $y = x$ (i.e. to the bisector of the angle formed by axes). Let this line meet the parabola again at U. Then projections of the segment UV have length 1.

Second solution. The distance between the origin and the intersection of the graph with y–axis is $|b|$. If the graph does not contain the origin ($b \neq 0$), we construct the points $(\pm b, 0)$. Raising perpendiculars to Ox at these points and marking their intersections with the graph, we construct segments of length $|b^2 \pm ab + b|$. We know the signs of the expressions $b^2 \pm ab + b$ (because we see on wich side of Ox the intersections lie), so we can construct a segment with length $\left|\frac{(b^2+ab+b)+(b^2-ab+b)}{2} - b\right| = b^2$. Now we can construct a right triangle ABC with legs $AC = |b|$ and $BC = b^2$, and a triangle $A'B'C'$ similar to ABC, where $B'C' = |b|$. In this triangle we have $A'C' = 1$.

If the graph contains the origin ($b = 0$), the segment between the common points of Ox and the parabola is $|a|$. Similarly to the previous case we construct a^2 and then 1.

6. Since $AB = BC$, the point B belongs to the radical axis of ω_1 and ω_2; this radical axis is their common tangent at X. Let $\angle ABX = 2\alpha$, $\angle CBX = 2\gamma$. Then



$$\angle BAX = \angle BXA = 90^\circ - \alpha,$$
$$\angle BCX = \angle BXC = 90^\circ - \gamma.$$

Note that B is the circumcentre of the triangle AXC, therefore $\angle CAX = \gamma$, $\angle ACX = \alpha$. Finally,

$$\angle AXY = \angle BAC = 90^\circ - \alpha - \gamma = \angle BCA = \angle ZXC,$$

the first and the last equations referring to the angle between a tangent and a chord. Now it is easily calculated that

$$\angle ZYX = 90^\circ - \alpha, \quad \angle YZX = 90^\circ - \gamma, \quad \angle YXB = \gamma, \quad \angle BXZ = \alpha.$$

Hence the line XB contains the point O. Furthermore, O_1O is the perpendicular bisector of YX, and $CX \perp YX$ (by angle counting). Thus $OO_1 \parallel XC$ and $BO = BA_1$. Similarly, $BO = BC_1$.

7. If a policeman shadows someone not trusted by the thief, he will never catch the thief; let us call such policeman *redundant*. We prove that the thief can use a greedy algorithm, dismissing every evening an accompilce to make more policemen redundant. On Day k, the thief can dismiss any of $2m-k+1$ accomplices, and each useful policeman can be made redundant in k ways. Therefore in this step the ratio of policemen which can be made redundant is at least $\frac{k}{2m-k+1}$. It remains to note that

$$\prod_{k=1}^{m}\Bigl(1 - \frac{k}{2m-k+1}\Bigr) = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-1} \cdot \frac{2m-5}{2m-2} \cdot \ldots \cdot \frac{1}{m+1}.$$

Multiplying the numerator and the denominator by

$$m! = 2^{-m} \cdot 2m \cdot (2m-2) \cdot \ldots \cdot 2,$$

we see that the product equals 2^{-m}.

Thus if the number of policemen is less than 2^m, all of them become redundant at some moment.

8. The answer is 2^n.

Lemma. Since the polynomials $(x+1)^{n+1}$ and x^{n+1} are coprime, the condition "the degrees of P and Q do not exceed n" determines P and Q uniquely.

Proof. Let

$$P_1(x)x^{n+1} + Q_1(x)(x+1)^{n+1} = 1 = P_2(x)x^{n+1} + Q_2(x)(x+1)^{n+1}.$$

Collect all expressions in the left hand side:

$$(P_1(x) - P_2(x))x^{n+1} + (Q_1(x) - Q_2(x))(x+1)^{n+1} = 0.$$

Therefore $(P_1(x)-P_2(x))x^{n+1}$ is divisible by $(x+1)^{n+1}$. Since polynomials x^{n+1} and $(x+1)^{n+1}$ are coprime, $P_1(x) - P_2(x)$ is divisible by x^{n+1}. But $\deg(P_1 - P_2) \leqslant n$, hence $P_1(x) - P_2(x) = 0$. Analogously $Q_1(x) = Q_2(x)$. $\square$

Put $x = -1 - y$ in the given identity:

$$Q(-1-y)(-1)^{n+1}y^{n+1} + P(-1-y)(-1)^{n+1}(y+1)^n = 1.$$

By lemma

$$P(x) = (-1)^{n+1}Q(-1-x), \quad Q(x) = (-1)^{n+1}P(-1-x).$$

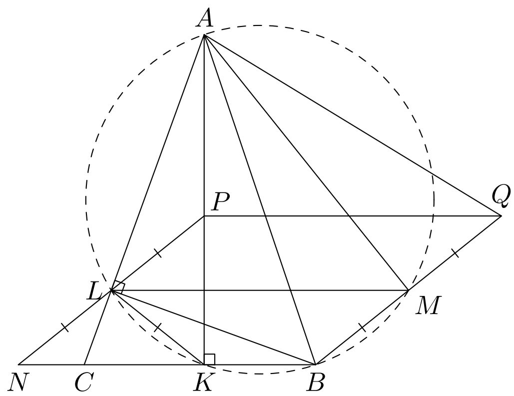
Put $x = -\frac{1}{2}$ and obtain

$$P\Big(-\frac{1}{2}\Big) = (-1)^{n+1}Q\Big(-\frac{1}{2}\Big) \quad \text{and} \quad P\Big(-\frac{1}{2}\Big)\cdot\Big(-\frac{1}{2}\Big)^{n+1} + Q\Big(-\frac{1}{2}\Big)\cdot\Big(\frac{1}{2}\Big)^{n+1} = 1,$$

i. e. $Q(-\frac{1}{2}) = 2^n$.

Junior League

6. Let N be the intersection of lines BC and LP, and M be the midpoint of BQ. In the right triangle NKP $KL = LP$, therefore KL is the median, and $NL = LP$. The line LM is the midsegment of the parallelogram $NPQB$, and is therefore parallel to its base. Note that A, B, K, L belong to the same circle. This circle also contains M, since $MBKL$ is an iscosceles trapezoid. Thus $AM \perp BQ$, that is, the triangle ABQ is iscosceles, and $\angle ABQ = \angle AQB$. It remains to prove that $\angle ABQ = \angle ACB$. This is equivalent to $\angle CBQ = 180° - \angle CAB$. But this is true:



$$\angle CBQ = 180° - \angle PNK = 180° - \angle CKL = 180° - \angle CAB.$$

7. The answer is 171.

Let $N = \overline{a_n a_{n-1} \ldots a_1 a_0}$. Then

$$a_n 10^n + a_{n-1}10^{n-1} + \ldots + a_0 =$$
$$= a_n(a_{n-1}+1)\ldots(a_0+1) + a_{n-1}(a_{n-2}+1)\ldots(a_0+1) + \ldots + a_1(a_0+1) + a_0.$$

Each term of the LHS is greater or equal to the respective term of the RHS, the equality being possible only when $a_{n-1} = a_{n-2} = \ldots = a_0 = 9$. The digit a_n can be any but 0. In total we have $9 \cdot 19 = 171$ numbers.

8. We shall use Cartesian coordinates to describe the marked points; for instance, the angles of the strip are $(0,0)$, $(n,0)$, $(n,1)$, $(0,1)$. Let $f(n)$ be the number of partitions of $1 \times n$ strip.

We prove by induction that 2^n divides $f(n)$. *Induction base:* $n=1$, $f(1)=2$.

Induction step. Consider some partition P of $1 \times (n-1)$ strip. We assign to it partitions of $1 \times n$ strip in the following way. We move the upper side of $1 \times (n-1)$ strip by 1 to the right, thus moving upper ends of all the segments and obtaining a partition of $1 \times n$ strip which satisfies the condition and, moreover, contains no vertical segments. In the partition of $1 \times n$ strip thus obtained we may draw, or we may not draw the segments from $(0,0)$ to $(1,1)$ and from $(n-1,0)$ to $(n,1)$ (when the upper side was moved these segments resulted from vertical sides of the original strip; they were not present in the original partition). Thus we have four partitions of $1 \times n$ strip without vertical segments assigned to each partition of $1 \times (n-1)$ strip. It is easily verified that the inverse transform assign a partition of $1 \times (n-1)$ strip to each partition of $1 \times n$ strip without vertical segments. So there exist exactly $4f(n-1)$ partitions of $1 \times n$ strip without vertical segments. By induction hypothesis, 2^n divides $4f(n-1)$.

Now we shall count the number of partitions where the leftmost vertical segment has abscissa k. This segment divides $1 \times n$ strip into $1 \times k$ strip without vertical segments and $1 \times (n-k)$ strip that may be partitioned in any way. Similar argument shows that the number of such partitions is $4f(k-1)f(n-k)$ for $k > 1$. When $k=1$ the number of such partitions is $2f(n-1)$. Thus for each k we get a number of partitions divisible by 2^n.