REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD
"TUYMAADA-2020"
(mathematics)

First day

Yakutsk 2020

The booklet contains the problems of XXII International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by S. L. Berlov, A. S. Golovanov, S. V. Ivanov, K. P. Kokhas, A. S. Kuznetsov, N. Y. Vlasova. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

1. Does the system of equations

$$\begin{cases} x_1 + x_2 = y_1 + y_2 + y_3 + y_4 \\ x_1^2 + x_2^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 \\ x_1^3 + x_2^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3 \end{cases}$$

admit a solution in integers such that the absolute value of each of these integers is greater than 2020?

(*A. Choudhry*)

2. Given are positive real numbers $a_1, \ldots, a_n$. Let

$$m = \min\left(a_1 + \frac{1}{a_2},\ a_2 + \frac{1}{a_3},\ \ldots,\ a_{n-1} + \frac{1}{a_n},\ a_n + \frac{1}{a_1}\right).$$

Prove the inequality

$$\sqrt[n]{a_1 a_2 \ldots a_n} + \frac{1}{\sqrt[n]{a_1 a_2 \ldots a_n}} \geqslant m\,.$$

(*I. Privalov*)

3. Points D and E lie on the lines BC and AC respectively so that B is between C and D, C is between A and E, $BC = BD$, and $\angle BAD = \angle CDE$. It is known that the ratio of the perimeters of the triangles ABC and ADE is 2. Find the ratio of the areas of these triangles.

(*A. Kuznetsov*)

4. For each positive integer k let $g(k)$ be the maximum possible number of points in the plane such that pairwise distances between these points have only k different values. Prove that there exists k such that $g(k) > 2k + 2020$.

(*Folklore*)

Junior League

1. For each positive integer m let t_m be the smallest positive integer not dividing m. Prove that there are infinitely many positive integers which can not be represented in the form $m + t_m$.

(*A. Golovanov*)

2. All non-zero coefficients of the polynomial $f(x)$ equal 1, while the sum of the coefficients is 20. Is it possible that thirteen coefficients of $f^2(x)$ equal 9?

(*S. Ivanov, K. Kokhas*)

3. Each edge of a complete graph with 101 vertices is marked with 1 or -1. It is known that absolute value of the sum of numbers on all the edges is less than 150. Prove that the graph contains a path visiting each vertex exactly once such that the sum of numbers on all edges of this path is zero.

(*Y. Caro, A. Hansberg, J. Lauri, C. Zarb*)

4. See problem 3 of Senior league.

SOLUTIONS

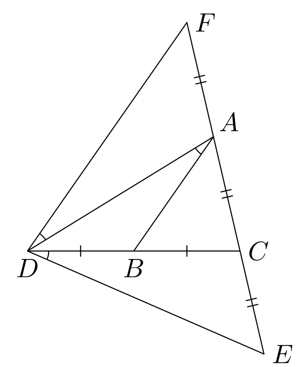
Senior League

1. The answer is yes.

Any Pythagorean triple $a^2+b^2=c^2$ gives a solution $(c,-c,a,-a,b,-b)$. It is enough to take $a=2020\cdot 3$, $b=2020\cdot 4$, $c=2020\cdot 5$.

2. Let $\sqrt[n]{a_1a_2\dots a_n}=a$. There exists $1\leqslant i\leqslant n$ such that $a_i\leqslant a$, $a_{i+1}\geqslant a$ (indices are modulo n). Indeed, we can find some $a_j\leqslant a$ and increase the index until we meet a number greater or equal to a. Obviously $m\leqslant a_i+\frac{1}{a_{i+1}}\leqslant a+\frac{1}{a}$, q.e.d.

3. The answer is 4.



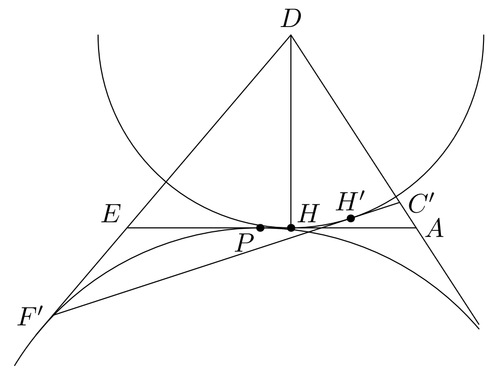
Since the triangle ADE contains the triangle ABC, $2P(ABC)=P(ADE)$. Let point F on the line AC be the reflection of C across A. Then $P(CDF)=2P(ABC)=P(ADE)$, and $\angle ADF=\angle BAD=\angle CDE$. Thus the triangles CDF and ADE have equal perimeters, angles at D and altitudes drawn from D. It follows from the lemma (see below) that these triangles are equal. Then $S(ADE)=S(CDF)=4S(ABC)$.

Lemma. Let the triangles ADE and CDF have equal angles at D, altitudes drawn from D, and perimeters. Then the triangles ADE and CDF are equal.

Proof 1. Assume that in a triangle the angle at D is α, the altitude drawn from D is h_a and the semiperimeter is p. If a is the side opposite D, r the inradius, and S the area, we have $(p-a)/r=\cot\frac{\alpha}{2}$ and $a/r=2S/rh_a=2p/h_a$. It follows that $p/r=\cot\frac{\alpha}{2}+2p/h_a$, thus r, and therefore $a=2pr/h_a$ are determined by p, α, and h_a. It remains to note that a side a, the opposite angle α, and the altitude h_a, in their turn, determine the triangle: the vertex opposite a belongs to the line parallel to a at the distance h_a from it, and to the arc determined by a and α.

Proof 2. Assume without loss of generality that $AD\leqslant DE$ and $CD\leqslant DF$. Let C' and F' be the points on rays DA and DE respectively such that $DC'=DC$ and $DF'=DF$ (then $\Delta C'DF'=\Delta CDF$). Let H and

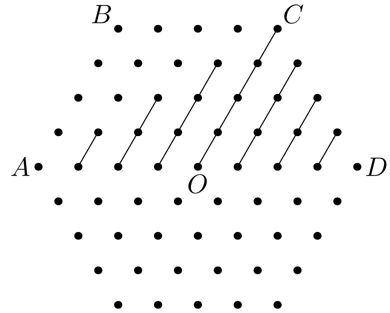
H' be the bases of the altitudes drawn from D in the triangles ADE and $C'DF'$ respectively. Our assumptions on side lengths ensure that H, H', and ray DA belong to the same half-plane bounded by the bisector of D. Let, again without loss of generality, ray DH be nearer the bisector than ray DH' (if these rays coincide, so do the triangles). We can now prove that $P(ADE) < P(C'DF')$. The points H and H' belong to the circle with centre D and radius DH, while AE and $C'F'$ touch this circle at H and H'. The points H' and A are on the same side of DH, therefore the tangent at H' intersects the ray HA (the point of intersection belongs to the bisector of HDH'). The point where the excircle of ADE touches AE lies on the ray HE (since $AD \leqslant DE$). We denote this point by P.



We have seen that P and D are from the same side of $C'F'$. But the excircle of $C'DF'$ is on the other side. It follows that the excircle of $C'DF'$ is greater, and therefore the perimeter of $C'DF'$ is greater (both circles are inscribed in the angle D, and the tangents from D equal the semiperimeters of the respective triangles).

4. An example of desired configuration is provided by points of the triangular lattice contained in a regular hexagon with side $n-1$ (border included). Each side of such hexagon contains exactly n lattice points; it is easily to find that there are $3n^2-3n+1$ lattice points in the entire hexagon.

We need an upper estimate on the number of pairwise distances between lattice points in the hexagon. Every segment connecting two lattice points in the hexagon (and thus providing one of the distances in question) can be translated so that one of its ends is a vertex of the hexagon. Let us denote this vertex by A, the other end of the translated segment by X, and the centre of the hexagon by O. The longest diagonal of the hexagon drawn from A divides it into two equal quadrilaterals; the segment AX belongs to one of these quadrilateral, say $ABCD$. This quadrilateral consists of parallelogram $ABCO$ and regular triangle COD. If X belongs to the parallelogram, we can assume (since we need only the length of AX) that X lies in the triangle ACO. Clearly this triangle contains



$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

lattice points (counting along the lines parallel to CO), these points being all possoble positions of X. If X belongs to COD, we have

$$1 + 2 + 3 + \ldots + (n-1) = \frac{n(n-1)}{2}$$

possible positions (lattice points on OC are counted above). In total, we have $\frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2$ possible positions of X.

Thus $N = 3n^2 - 3n + 1$ lattice points in the hexagon define at most $k = n^2$ possible distances. Obviously the inequality $N > 2k + 2020$ holds for large n.

Junior League

1. For instance, powers of 2 can not be represented in the desired form. Indeed, if

$$2^k = m + t_m,$$

then the number m must be even (since $t_m = 2$ for odd m and then the RHS is odd). Let $m = 2^n \cdot d$, where d is odd, $n < k$. If $m + t_m$ is a power of 2, the number t_m is divisible by 2^n (and greater than 2^n, because 2^n divides m). Then, however, $t_m = 2^{n+1}$ and $m + t_m = 2^n(2 + d) \neq 2^k$.

2. Answer: this is impossible.

Suppose there exists such a polynomial

$$f(x) = x^{d_1} + x^{d_2} + \ldots + x^{d_{20}}.$$

Then $f^2(x) = \sum x^{2d_i} + \sum\limits_{i<j} 2x^{d_i+d_j}$. It follows that only monomials of degrees $2d_\ell$ can have odd coefficients. Let the coefficient of x^{2d_ℓ} is 9. Then there exist four pairs $i < j$ such that $d_i + d_j = 2d_\ell$. For each of these pairs $d_i > d_\ell > d_j$. Thus there exist at least four exponents d_i greater than d_ℓ, and at least four exponents d_j less than d_ℓ. This means that $5 \leqslant \ell \leqslant 16$, that is, there is at most 12 such coefficients.

3. To solve the problem it suffices to prove the existence of a Hamiltonian cycle (i. e., a cycle visiting each vertex exactly once) such that the sum of numbers on all its edges is ± 1. Such a cycle contains an edge marked with $+1$ and an edge marked with -1. Deleting one of these we obtain the desired path.

Now we prove that such cycle exists. Let us call the sum of a cycle the sum of all the numbers on its edges. Clearly a cycle of odd length has an odd sum. The edges of our graph can be distributed between 50 Hamiltonian cycles. (For instance, we can arrange the vertices on a circle and go to each k-th vertex. For every $1 \leqslant k \leqslant 50$ we get a Hamiltonian cycle: since 101 is prime, we can not return to any vertex before visiting all the others.) Assume that none of these cycles has sum ± 1. All these 50 sums can not be simultaneously greater or equal to 3, or simultaneously lesser or equal to -3, since otherwise the absolute value of all the numbers of edges would be at least 150. Therefore there exists a cycle C_1 with sum less than 3; by our assumption this sum is at most -3. Similarly, there exists a cycle C_2 with sum greater or equal to 3.

The cycle C_2 can be obtained from the cycle C_1 by a sequence of operations exchanging two consecutive vertices of a cycle. Each operation changes the sum of the cycle at most by 4 (because it replaces two edges by two other edges). Therefore at some moment we had an "intermediate" cycle with sum ± 1.