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MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD
"TUYMAADA-2021"
(mathematics)

Second day

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The booklet contains the problems of XXVIII International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by M. A. Antipov, S. L. Berlov, A. S. Golovanov, S. V. Ivanov, K. P. Kokhas, A. S. Kuznetsov, E. M. Lopatin, D. Y. Shiryaev, N. Y. Vlasova. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

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Senior league

5. Sines of three acute angles form an arithmetic progression, while the cosines of these angles form a geometric progression. Prove that all three angles are equal.

(S. Berlov)

6. In a $n \times n$ table ($n > 1$) k unit squares are marked. One wants to rearrange rows and columns so that all the marked unit squares are above the main diagonal or on it. For what maximum k is it always possible?

(M. Antipov)

7. An acute triangle ABC is given, $AC \neq BC$. The altitudes drawn from A and B meet at H and intersect the external bisector of the angle C at Y and X respectively. The external bisector of the angle AHB meets the segments AX and BY at P and Q respectively. If $PX = QY$, prove that $AP + BQ \geqslant 2CH$.

(E. Lopatin, D. Shiryaev)

8. In a sequence P_n of quadratic trinomials each trinomial, starting with the third, is the sum of the two preceding trinomials. The first two trinomials do not have common roots. Is it possible that P_n has an integral root for each n?

(A. Golovanov)

Junior League

5. In a 100×100 table 110 unit squares are marked. Is it always possible to rearrange rows and columns so that all the marked unit squares are above the main diagonal or on it?

(M. Antipov)

6. Given are real $y > 1$ and positive integer $n \leqslant y^{50}$ such that all prime divisors of n do not exceed y. Prove that n is a product of 99 positive integer factors (not necessarily primes) not exceeding y.

(G. Martin, A. Parvardi)

7. A pile contains 2021^{2021} stones. In a move any pile can be divided into two piles so that the numbers of stones in them differ by a power of 2 with non-negative integer exponent. After some move it turned out that the number of stones in each pile is a power of 2 with non-negative integer exponent. Prove that the number of moves performed was even.

(M. Antipov)

8. See problem 7 of Senior league.

SOLUTIONS

Senior League

5. *First solution.* Assume that angles $\alpha \le \beta \le \gamma$ satisfy the condition of the problem. It is clear that if two of the angles are equal then the arithmetic progression is constant and then all three angles are equal. So we may assume that both inequalities are strict. The sine increases and the cosine decreases on the segment $[0°, 90°]$. Therefore $\sin\beta$ and $\cos\beta$ are middle terms in the respective progressions. Rewrite the condition as follows:

$$2\sin\beta = \sin\alpha + \sin\gamma, \quad \cos^2\beta = \cos\alpha \cdot \cos\gamma.$$

Squaring the first equality and adding the second multiplied by 4, we get

$$4 = 4\sin^2\beta + 4\cos^2\beta = \sin^2\alpha + 2\sin\alpha\sin\gamma + \sin^2\gamma + 4\cos\alpha\cos\gamma \leqslant \\ \leqslant 2\sin^2\alpha + 2\sin^2\gamma + 2\cos^2\alpha + 2\cos^2\gamma = 4.$$

Then the central inequality should be in fact an equality, which is possible only when $\sin\alpha = \sin\gamma$, a contradiction.

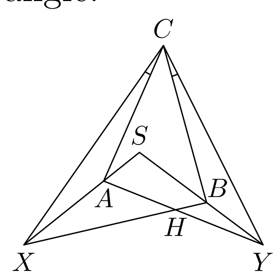
Second solution. Let $f(x) = \ln\cos\arcsin x$, $x \in [0, 1)$. Assume that the angles are different and their sines x_1, x_2, x_3 form an arithmetic progression. Then the numbers $y_i = f(x_i)$ in the same order also form an arithmetic progression (because f is monotone). Hence points (x_1, y_1), (x_2, y_2), (x_3, y_3) of the Cartesian plane belong to the same line, and these points belong also to the graph of function $y = f(x)$. Therefore some line intersects the graph of $f(x)$ at three points. This is impossible because $f(x)$ is concave on the segment $[0, 1)$. Indeed, $f(x) = \frac{1}{2}\ln(1-x^2)$, $f'(x) = \frac{x}{x^2-1}$, $f''(x) = -\frac{x^2+1}{(x^2-1)^2} < 0$.

6. The answer is $n + 1$.

We will prove by induction on n that if $k \leqslant n + 1$ unit squares are marked, one can always rearrange rows and columns so that all the marked squares are not below the main diagonal. The base $n = 2$ is trivial. To prove the inductive step, let us choose a row R containing at least two marked squares (if there is no such row, then $k \le n$, and we choose any row containing at least one marked square). Then we choose a column C containing no marked squares except possibly the square at the intersection with R. Such a column exist, otherwise there are at least n marked squares outside R, contrary to its choice. Move R so that it becomes the topmost row of the table and C to become the leftmost column. To the remaining part of the table we can apply the inductive hypothesis.

There is an example of $k = n+2$ marked squares which are impossible to rearrange in the desired manner. Let four marked squares form 2×2 square in the upper left corner of the table, and all the remaining squares of the main diagonal are also marked. We will prove the impossibility by induction on n. The base case, $n = 2$, is obvious. Suppose there is a desired rearrangement for some $n > 2$. Choose a row containing one marked square; the column containing this square does not contain other marked squares. If this square is not on the main diagonal, that is, to the right of it, the column containing it can be moved left so that the marked square gets on the main diagonal; all the other columns either remain on their places or move right, and the condition is not violated. When the marked square is on the main diagonal, it can be removed to apply the inductive hypothesis.

7. We begin by reminding the reader one definition and one lemma. Two lines passing through the vertex of an angle are called *isogonal* when they are symmetric with respect to the bisector of this angle.



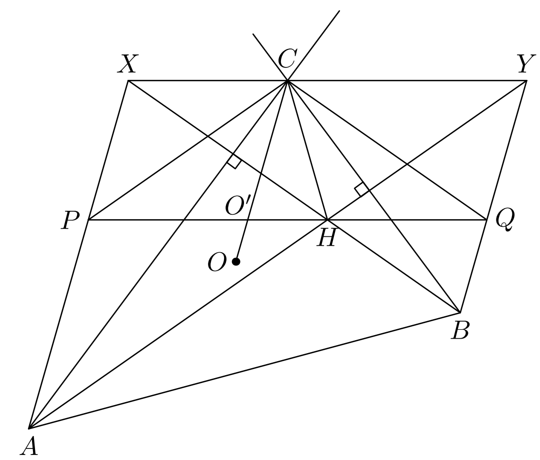
L e m m a (Isogonal Line Lemma). Let lines CA and CB be isogonal with respect to angle XCY, lines XA and YB intersect at point S, and lines XB and YA intersect at point H. Then lines CS and CH are also isogonal.

Let us apply this lemma to the problem. Let O be the circumcentre of ABC. Then $\angle ACO = \angle BCH$, i. e. lines CO and CH are symmetric with respect to the bisector of angle ACB, and therefore they are isogonal with respect to angle XCY.

We will prove that lines AX, BY, CO have common point (possibly at infinity, in which case the three lines are parallel) regardless of the condition $PX = QY$. Indeed, CA and CB are isogonal with respect to angle XCY. Then by Isogonal Line Lemma, CH and the line connecting C with the intersection of XA and YB are also isogonal, and we are done.

Now we prove that lines AX, BY, CO are parallel. By the previous claim it suffces to check that AX and BY are parallel. Let $\angle ACB = \gamma$. Then $\angle YCB = \angle XCA = 90° - \gamma/2$, $\angle AHB = 180° - \gamma$, and $\angle CYH =$

$\angle CXH = \angle YHQ = \gamma/2$. Therefore, $YX \parallel PQ$. Since $YQ = XP$, $YXPQ$ is a parallelogram or an isosceles trapezoid.



Assume that $YXPQ$ is an isosceles trapezoid. Then $\angle QYX = \angle YXP$, and, since $\angle CYH = \angle CXH$, we have $\angle BYA = \angle BXA$. It follows that the quadrilateral $YXAB$ is cyclic, and $YB = XA$ (the arcs subtending these chords are equal). Therefore $YXAB$ is an isosceles trapezoid (or rectangle) and $YX \parallel AB$. But this means that the external bisector of angle C in triangle ABC is parallel to its base AB. This is possible only when $AC = BC$, contradiction.

Thus $YXPQ$ is a parallelogram, lines AX and BY are parallel (hence they are parallel to CO). Let O' be the intersection point of PQ and CO. $PXCO'$ is a parallelogram, $PXCH$ is an isosceles trapezoid (by the same isogonality with respect to angle XCY). Thus $PX = QY = CH$ and $\angle CYH = \angle HXC = \angle XCP$. Therefore $CP \parallel AY$ and similarly $CQ \parallel BX$, thus

$$\frac{AP + BQ}{CH} = \frac{AP}{PX} + \frac{BQ}{QY} = \frac{CY}{CX} + \frac{CX}{CY} \geqslant 2.$$

8. It is easily proved by induction that

$$P_{n+2}(x) = F_n P_1(x) + F_{n+1} P_2(x), \qquad (*)$$

for $n \geqslant 0$, where F_n is n-th Fibonacci number. Let x_n be an integer root of P_n. Setting $x = x_{n+2}$ in $(*)$ we have

$$F_n P_1(x_{n+2}) + F_{n+1} P_2(x_{n+2}) = 0.$$

Since P_1 and P_2 have no common roots, it follows immediately that x_{n+2} is not a root of either P_1 or P_2. Then

$$-\frac{F_{n+1}}{F_n} = \frac{P_1(x_{n+2})}{P_2(x_{n+2})}. \qquad (**)$$

It is well known that $(F_n, F_{n+1}) = 1$, that is, fractions $\frac{F_{n+1}}{F_n}$ are irreducible and, since $F_n < F_{n+1}$ for $n > 1$, distinct. It follows that all x_n are distinct, and (since they are integers) $|x_n| \to +\infty$ when $n \to +\infty$. It is at least as well, or even a little better, known that $\lim\limits_{n\to+\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$.

Then, taking the limit in (∗∗) as n tends to infinity, we see that the ratio of the leading coefficient of P_1 and the leading coefficient of P_2 is $-\frac{1+\sqrt{5}}{2}$.

We will prove that in fact that ratio must be rational. Without loss of generality assume that the leading coefficient of P_1 is 1 (dividing all the coefficients of P_1 and P_2 by the same number we do not change the roots of P_n). We will call a function *quadratic rational* if it is a ratio of two quadratic trinomials without common roots; clearly we can assume that the leading coefficient of the trinomial in the numerator is 1.

L e m m a. Let q_i, r_i, $1 \leqslant i \leqslant 5$ be real numbers, and q_i pairwise distinct. Then there is at most one quadratic rational function f satisfying $f(q_i) = r_i$, $i = 1, \ldots, 5$.

Indeed, if $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$ are two such functions, the numerator of their difference, i.e. the polynomial $u_1v_2 - u_2v_1$, has five roots. It remains to note that the degree of this (non-zero) polynomial does ot exceed 4.

Back to the problem, note that the quadratic rational function $P_1(x)/P_2(x)$ has rational values $-F_{k+1}/F_k$ at five integer points x_{k+2}, $k = 1, 2, 3, 4, 5$ (in fact, there are infinitely many such points, but five is enough for us). It follows from the Lemma that such function is unique if it exists at all. On the other hand, five conditions $F_kP_1(x_{k+2}) + F_{k+1}P_2(x_{k+2}) = 0$ are a system of five linear equations with five variables, namely, the coefficients of P_1, P_2. We have seen that this system admits a unique solution, which must be therefore rational, since the coefficients are rational and the variables can be expressed through them by elimination (or Cramer's rule).

Junior League

5. The answer is no.

It is impossible to rearrange rows and columns if the set of marked squares contains 102 squares described in the example for problem 6 of Senior league ($n = 100$, $k = 102$).

6. Let $p_1 \geq p_2 \geq \ldots \geq p_r$ be the prime divisors of n. We use greedy algorithm to distribute these primes in 99 piles: every next p_s goes to any pile with minimum product of numbers in it. We denote by b_i the product of primes already put in the i-th pile (in the beginning $b_i = 1$ for all i). If a prime p_s cannot be added to any pile in its turn, we have $b_i > y/p_s$ for all i, and

$$(y/p_s)^{99}p_s < b_1b_2\ldots b_{99}p_s \leq n \leq y^{50}.$$

it follows from this inequality that $y^{49} < p_s^{98}$, that is, $p_s > \sqrt{y}$. But $b_i \geq p_s$ for all i (every pile already contains at least one prime not less than p_s), and $n \geq b_1b_2\ldots b_{99}p_s \geq p_s^{100} > y^{50}$, a contradiction.

7. For brevity's sake we call the remainder left by the number of stones in a pile upon division by 3 merely *the remainder of a pile* (probably reconstructing the history of this notion). Consider the remainders of two new piles obtained by splitting a pile in two. Since 3 cannot divide a power of 2, the new piles have different remainders. Then from a pile with remainder 0 we can obtain only piles with remainders 1 and 2; from a pile with remainder 1 we can obtain only piles with remainders 0 and 1; finally, from a pile with remainder 2 we can obtain only piles with remainders 0 and 2; It is immediately seen that each operation changes the number of zero remainders by 1. Since there were no piles with remainder 0 either in the beginning or in the end of the process, the number of operations was even.

8. See Problem 7, Senior league.