REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD
"TUYMAADA-2021"
(mathematics)

First day

Yakutsk 2021

The booklet contains the problems of XXVIII International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by S. L. Berlov, A. S. Golovanov, V. I. Frank, K. P. Kokhas, A. S. Kuznetsov, N. Y. Vlasova. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

1. Polynomials F and G satisfy

$$F(F(x)) > F(G(x)) > G(G(x))$$

for all real x. Prove that $F(x) > G(x)$ for all real x.

(*V. Frank*)

2. In trapezoid $ABCD$, M is the midpoint of base AD. Point E lies on the segment BM. It is known that $\angle ADB = \angle MAE = \angle BMC$. Prove that the triangle BCE is isosceles.

(*A. Kuznetsov*)

3. Positive real numbers $a_1, \ldots, a_k, b_1, \ldots, b_k$ are given. Let $A = \sum_{i=1}^{k} a_i$, $B = \sum_{i=1}^{k} b_i$. Prove the inequality

$$\left(\sum_{i=1}^{k} \frac{a_i b_i}{a_i B + b_i A} - 1\right)^2 \geqslant \sum_{i=1}^{k} \frac{a_i^2}{a_i B + b_i A} \cdot \sum_{i=1}^{k} \frac{b_i^2}{a_i B + b_i A}.$$

(*F .Dong, J. Ge*)

4. An $n \times n$ square (n is a positive integer) consists of n^2 unit squares. *A monotonous path* in this square is a path of length $2n$ beginning in the left lower corner of the square, ending in its right upper corner and going along the sides of unit squares.

For each k, $0 \leq k < 2n-1$, let S_k be the set of all the monotonous paths such that the number of unit squares lying below the path leaves remainder k upon division by $2n-1$. Prove that all S_k contain equal number of elements.

(*M.Just, M. Schneider*)

Junior League

1. Quadratic trinomials F and G satisfy

$$F(F(x)) > F(G(x)) > G(G(x))$$

for all real x. Prove that $F(x) > G(x)$ for all real x.

2. The bisector of angle B of a parallelogram $ABCD$ meets its diagonal AC at E, and the external bisector of angle B meets line AD at F. M is the midpoint of BE. Prove that $CM \parallel EF$.

(*A. Kuznetsov*)

3. For n distinct positive integers all their $n(n-1)/2$ pairwise sums are considered. For each of these sums Ivan has written on the board the number of original integers which are less than that sum and divide it. What is the maximum possible sum of the numbers written by Ivan?

(*S. Berlov*)

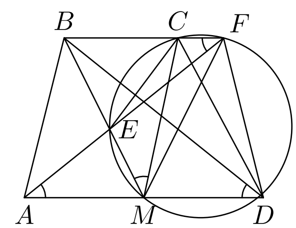
4. Some manors of Lipshire county are connected by roads. The inhabitants of manors connected by a road are called *neighbours*. Is it always possible to settle in each manor a knight (who always tells truth) or a liar (who always lies) so that every inhabitant can say "The number of liars among my neighbours is at least twice the number of knights"?

(*S. Berlov*)

SOLUTIONS

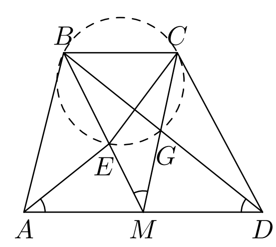
Senior League

1. It follows from the right inequality that $F(x) > G(x)$ on all the range of $G(x)$. Suppose the inequality does not hold for all x. Since $F(x) - G(x)$ is continuous, there is a point x_0 with $F(x_0) = G(x_0)$. This, however, implies $F(F(x_0)) = F(G(x_0))$ and contradicts the left inequality.



2. First solution. Extend AE to meet BC at F. Then $ABFD$ is an isosceles trapezoid (since the bases are parallel and the diagonals form equal angles with the base), the triangle BMF is isosceles (since an isosceles trapezoid is symmetric across the perpendicular bisector of its bases), and the quadrilateral $MECF$ is cyclic (since $\angle EMC = \angle BFE = \angle EAD$). Thus $\angle BCE = \angle FME$, i.e. triangles BCE and BMF are similar, and BCE is therefore isosceles.

Remark. Strictly speaking, the solution depends on the position of point F. However, all the angle equalities hold independently of F being to the left or to the right of C. And if F coincides with C (that is, the trapezoid $ABCD$ is itself isosceles), the equality $\angle BCE = \angle EAD = \angle EMC$ follows immediately from parallelism, and then the solution follows from the same similarity.



Second solution. Let G be the intersection point of BD and MC. Then $\angle BCM = \angle GMD = \angle MEA$ (the first equality follows from parallelism, and the second is true since the angle in question are supplementary to $\angle EMA$ and $\angle BMC = \angle EAM$, respectively). Thus triangles DGM and MBC are similar and $DM/MC = MG/BC$. By the same argument AME and MBC are similar and $AM/MB = ME/BC$. It follows that $MG \cdot MC = DM \cdot BC = AM \cdot BC = ME \cdot MB$, that is, $BCGE$ is cyclic. Then $\angle BEC = \angle BGC = \angle MGD = \angle MBC$ (the last equality holds by similarity of triangles DGM and MBC), and the triangle BCE is isosceles.

3. This inequality is in fact an equality. Let, for the sake of brevity, $W_i = a_iB + b_iA$. Note that

$$A\sum_{i=1}^{k}\frac{a_ib_i}{W_i} + B\sum_{i=1}^{k}\frac{a_i^2}{W_i} = \sum_{i=1}^{k}\frac{a_i(a_iB + b_iA)}{a_iB + b_iA} = \sum_{i=1}^{k}a_i = A.$$

Therefore

$$\sum_{i=1}^{k}\frac{a_i^2}{W_i} = \frac{A}{B}\left(1 - \sum_{i=1}^{k}\frac{a_ib_i}{W_i}\right)$$

and similarly

$$\sum_{i=1}^{k}\frac{b_i^2}{W_i} = \frac{B}{A}\left(1 - \sum_{i=1}^{k}\frac{a_ib_i}{W_i}\right).$$

It remains to multiply the two equations.

4. It suffices to prove that the set of all monotonous paths can be partitioned into parts containing $2n-1$ paths each, one in each S_k. Moreover, we will separately partition thus the set of monotonous paths beginning with a horizontal segment, and the set of monotonous paths beginning with a vertical segment

Let $ABCD$ be our $n \times n$ square, and a path going from A to C begins with (say, horizontal) unit segment AF and ends with unit segment LC (see Fig. 1–7).

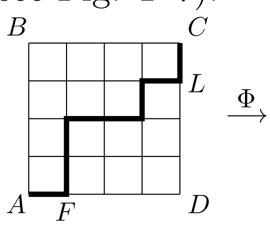


Fig. 1

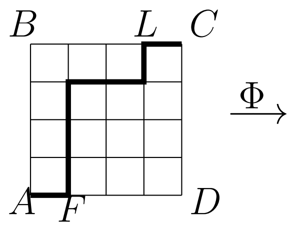


Fig. 2

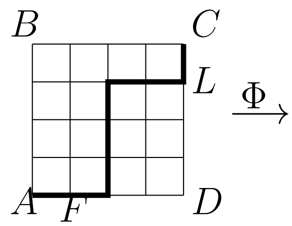


Fig. 3

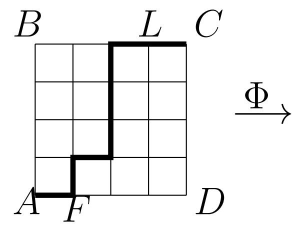


Fig. 4

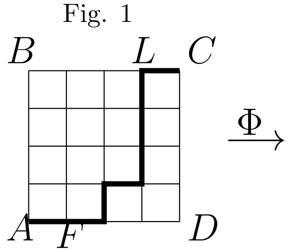


Fig. 5

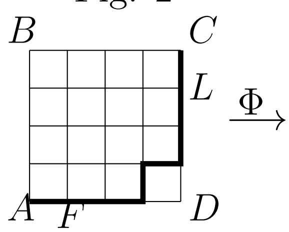


Fig. 6

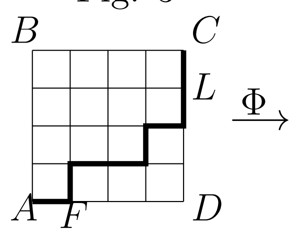


Fig. 7

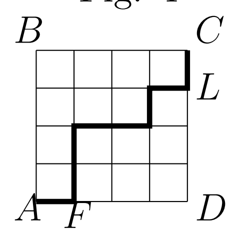


Fig. 1

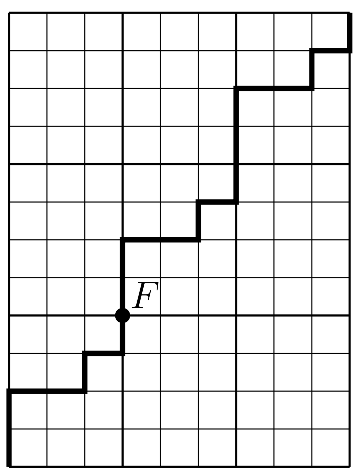
The transformation Φ, which we apply to paths to obtain new ones, acts as follows. We start with removing the segment LC. If LC was vertical, we insert a vertical unit segment right after AF and move the part FL of the path by 1 upwards (thus obtaining Fig. 2 from Fig. 1, Fig. 4 from Fig. 3 and so on). If LC was horizontal (see, for example, Fig. 2, 4, 5), the transformation inserts a horizontal unit segment after AF, and moves the part FL of the path by 1 to the right.

This transformation obviously either increases the number of unit squares under the path by $n-1$ or decreases ot by n. Either way it adds $n-1$ to that number modulo $2n-1$. Since $n-1$ and $2n-1$ are coprime, for each path γ all the paths

$$\gamma, \quad \Phi(\gamma), \quad \Phi(\Phi(\gamma)), \quad \dots \quad \underbrace{\Phi(\dots\Phi}_{2n-2\text{ times}}(\gamma))$$

are pairwise distinct (the numbers of unit squares under them are different modulo $2n-1$). The numbers of unit squares under these paths leave all possible remainders upon division by $2n-1$.

There is another interpretation of the above transformation. Let us partition the plane into $n\times(n-1)$ rectangles, choose a diagonal formed by these rectangles and draw the part F–C of the path under consideration in each of them. Every time when the transformation is performed, the point F is moved along the path by a (vertical or horizontal) unit segment, and the entire $n\times n$ square, where we observe the result of the transformation Φ, is moved with it.

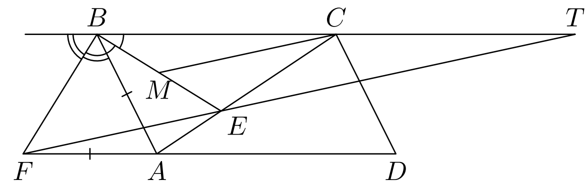


This interpretation makes clear that $2n-1$ applications of the transform return the path to original, and grouping each path γ with all its images under repeated transformations Φ splits the set of all paths into groups of $2n-1$ paths. We have seen that for each residue k modulo $2n-1$ each group contains exactly one path from the set S_k.

Junior League

1. See Problem 1, Senior league.

2. Note that $AB = AF$, since BAD is an exterior angle in the triangle FAB and equals



$$180° - \angle ABC = 2\angle FBA.$$

Let the lines FE and BC meet at T. Then the triangles FAE and TCE are similar, and $AF/CT = AE/CE$. It follows from angle bisector theorem that $AE/CE = AB/BC$. Therefore, $AF/CT = AB/BC = AF/BC$, whence $BC = CT$. Thus $CM \parallel ET$ as midline of triangle BET.

3. The answer is $\frac{(n-1)n(n+1)}{6}$.

First we prove by induction that the sum of the numbers on the board does not exceed $\frac{(n-1)n(n+1)}{6}$. For $n = 2$ it is obvious.

Assume that the statement is true for $n = k$; let us prove it for $n = k+1$. From a given set $a_1 < a_2 < \ldots < a_{k+1}$ we remove the largest number a_{k+1}. For the remaining nmbers the inductive hypothesis is true, i.e. the sum of the numbers written for them does not exceed $\frac{(k-1)k(k+1)}{6}$. For each $t \leqslant k$ the number a_t can divide at most $k+1-t$ sums $a_{k+1}+a_i$, since if a_t divides $a_{k+1}+a_i$ and $a_{k+1}+a_j$ for $i < j \leqslant t$, it also divides the difference $a_j - a_i < a_t$, which is impossible. As to the pairs $a_i + a_j$ divisible by a_{k+1}, they are not counted since $a_i+a_j < 2a_{k+1}$ and therefore $a_i + a_j = a_{k+1}$.

Thus when the number a_{k+1} is added to the set, the sum of the numbers added on the board does not exceed $\sum\limits_{i=1}^{k}(k+1-i) = \frac{k(k+1)}{2}$, making the total sum not greater than $\frac{(k-1)k(k+1)}{6} + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$.

The example of numbers giving that sum is provided by the numbers $1, 2, 4, \ldots, 2^{n-1}$. It is easy to see that every power of 2 adds exactly the number of pairs appearing in the above proof, thus giving the desired sum.

4. The answer is yes.

Let us settle gentlemen in the manors so that the sum of the number of pairs of neighbouring liars and twice the number of pairs of neighbouring knights is minimal, and among all the ways to do that we choose one in which the number of pairs of neighbouring knights is maximal. Suppose that this arrangement does not satisfy the condition. Two cases are possible:

1) There is a liar telling the truth. Let ℓ be the number of his lying neighbours and k the number of his neighbours-knights. Then $\ell \geqslant 2k$. Replace this liar by a knight. Then the number of pairs of neighbouring liars decreases by ℓ, twice the number of pairs of neighbouring knights increases by $2k$, that is, the sum does not increase. But the number of pairs of neighbouring knights increases, that is, the initial arrangement was not optimal, a contradiction.

2) There is a lying knight. Let ℓ be the number of his lying neighbours and k the number of his neighbours-knights, $\ell < 2k$. Similarly to 1), we can replace this knight by a liar. Then the number of pairs of neighbouring liars increases by ℓ, twice the number of pairs of neighbouring knights decreases by $2k$, and their sum decreases. This is again a contradiction with the choice of the arrangement.

Thus our arrangement satisfies all the requirements.