REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2019" (mathematics)

Second day

Yakutsk 2019

The booklet contains the problems of XXVI International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by M. A. Antipov, S. L. Berlov, A. S. Golovanov, K. P. Kokhas, A. S. Kuznetsov, N. Yu. Vlasova. Computer typesetting: K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

5. Do there exist six positive integers such that the greatest common divisor of every two of them is a prime number not exceeding 26, and every such prime number is the greatest common divisor of some two of these six numbers?

(*A. Golovanov*)

6. Prove that the product

$$(1^4 + 1^2 + 1)(2^4 + 2^2 + 1) \ldots (n^4 + n^2 + 1)$$

is not a perfect square for any positive integer n.

(*K. Gaitanas*)

7. Some N cells of a rectangular board are marked. Let a_i denote the number of marked cells in the i-th row and b_j denote the number of marked cells in the j-th column. Prove that

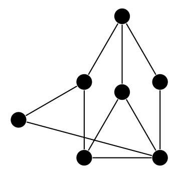
$$\prod_i a_i! \prod_j b_j! \leqslant N!$$

(*F. Petrov*)

8. In a triangle ABC the angle B is obtuse and $AB \neq BC$. O is the centre of the circumcircle ω of this triangle; N is the midpoint of the arc ABC. The circumcircle of BON meets the segment AC at X and Y. The rays BX and BY meet ω at the points $Z \neq B$ and $T \neq B$. Prove that the point symmetrical to N with respect to AC lies on the line ZT.

(*A. Kuznetsov*)

Junior League



5. Is it possible to draw in the plane the graph presented in the figure so that all the vertices are different points and all the edges are unit segments? (The segments can intersect at points different from vertices.)

(*A. Globus, H. Parshall*)

6. Do there exist six positive integers such that the greatest common divisor of every two of them is a prime number not exceeding 26, and every such prime number is the greatest common divisor of some two of these six numbers?

(*A. Golovanov*)

7. A circle ω touches the sides AB and BC of a triangle ABC and intersects its side AC at K. It is known that the tangent to ω at K is symmetrical to the line AC with respect to the line BK. What can be the difference $AK - CK$ if $AB = 9$ and $BC = 11$?

(*S. Berlov*)

8. Andy, Bess, Charley and Dick play on a 1000×1000 board. They make moves in turn: Andy first, then Bess, then Charley and finally Dick, after that Andy moves again and so on. At each move a player must paint several unpainted squares forming 2×1, 1×2, 1×3, or 3×1 rectangle. The player that cannot move loses. Prove that some three players can cooperate to make the fourth player lose.

(*S. Berlov, N. Vlasova*)

SOLUTIONS

Senior League

5. The answer is yes.

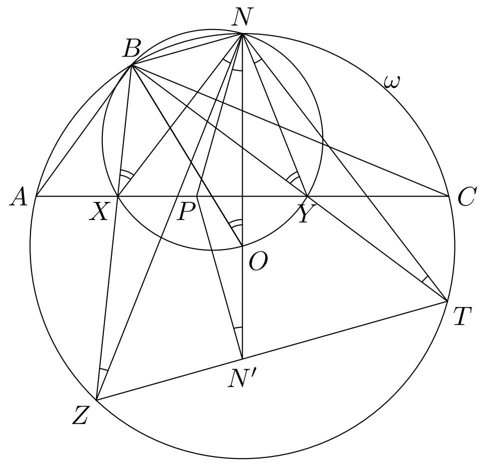
Solution. Here is an example of such numbers: $a = 2\cdot 5\cdot 7$, $b = 2\cdot 11\cdot 13\cdot 17$, $c = 2\cdot 3\cdot 19$, $d = 3\cdot 7\cdot 11\cdot 23$, $e = 3\cdot 5\cdot 13$, $f = 5\cdot 17\cdot 19\cdot 23$. We have $(a,b) = 2$, $(a,c) = 2$, $(a,d) = 7$, $(a,e) = 5$, $(a,f) = 5$, $(b,c) = 2$, $(b,d) = 11$, $(b,e) = 13$, $(b,f) = 17$, $(c,d) = 3$, $(c,e) = 3$, $(c,f) = 19$, $(d,e) = 3$, $(d,f) = 23$, $(e,f) = 5$.

6. Let $f(k) = k^2 - k + 1$; then $f(1) = 1$, $k^2 + k + 1 = f(k+1)$ and $k^4 + k^2 + 1 = f(k)f(k+1)$. Thus

$$(1^4+1^2+1)(2^4+2^2+1)\dots(n^4+n^2+1) =$$
$$= f(1)f(2)\cdot f(2)f(3)\cdot\ldots\cdot f(n)f(n+1) = (f(2)f(3)\dots f(n))^2 f(n+1).$$

But $f(n+1)$ is not a perfect square, since it lies between two consecutive squares: $n^2 < n^2+n+1 < (n+1)^2$. It follows that the product in question also cannot be a perfect square.

7. Let $A = \prod a_i!$, $B = \prod b_i!$. To prove the inequality $AB \leqslant N!$ it is enough to produce AB different ways to fill the squares by the numbers 1, 2, …, N (since the total number of such ways is obviously $N!$). At the first stage we put the numbers 1, 2, …, a_1 in the first row, the next a_2 numbers in the second row and so on; this can be done in A ways. Then we arbitrarily permute the numbers in columns (for every arrangement obtained at the first stage this can be done in B ways). All the arrangements obtained are different, since the first stage defines the set of numbers in each column and the second stage defines the order of these numbers.



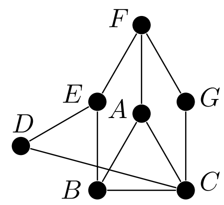
8. Assume, without loss of generality, that N lies on the arc BC of the circle ω that does not contain A. Let $\angle BON = 2\phi$, then $\angle BXN = 2\phi = \angle BYN$ since X and Y belong to the circumcircle of BON, and $\angle BZN = \phi = \angle BTN$ since Z and T belong to ω. It follows that the triangles NXZ and NYT are similar and isosceles, because they have equal angles

$\angle XZN = \angle YTN = \phi$ and external angles $\angle BXN = \angle BYN = 2\phi$. Thus there is a spiral similarity transforming the triangle NXZ to the triangle NYT. The dual homothety h with centre N that maps X to Z must also map the line XY to the line ZT.

Let N' be the point symmetrical to N with respect to AC, and P the point such that the triangles NXZ and NPN' are directly similar. Then on the one hand P lies on the line XY because XY is the perpendicular bisector of NN' and $PN = PN'$, while on the other hand $h(P) = N'$ by the definition of P. Since $h(XY) = ZT$, N' lies on ZT, q.e.d.

Junior League

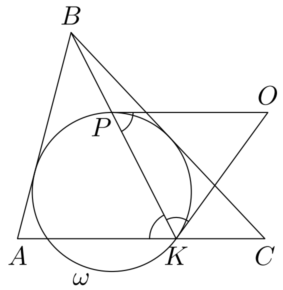
5. Let the vertices are denoted as shown in the figure. Suppose the graph is drawn in the plane so that the conditions are satisfied. Note that the vertices B and D lie at unit distance from E and C. Since they can not coincide, they lie in different half-planes with respect to EC, and together with E and C form a rhombus $DEBC$. Similarly, $ADEB$ and $CBEF$ are rhombi. Hence $\overrightarrow{CD} = \overrightarrow{BE} = \overrightarrow{AF} = \overrightarrow{CG}$, i. e. $\overrightarrow{CD} = \overrightarrow{CG}$ and $D = G$, a contradiction.



6. See Problem 5, Senior league.

7. The answer is 2.

Let P be the second point of intersection of the line BK and the circle ω, and O the point where the tangents to ω at K and P meet. Then $\angle KPO = \angle PKO$ since the triangle KOP is isosceles. Furthermore, $\angle AKP = \angle PKO$ because the lines KO and AC are symmetrical with respect to BK. Therefore $\angle KPO = \angle AKP$, and the lines PO and AC are parallel.



There is a homothety with centre O that maps PO to AC. This homothety maps ω to an excircle of ABC, and P to K. Thus K is the point where the excircle of the triangle ABC touches its side. It follows that $AK = p - AB$, $CK = p - BC$, where p is the semiperimeter of ABC, that is, $AK - KC = BC - AB = 2$.

8. F i r s t s o l u t i o n. We shall prove that A, B, and D can cooperate against C. In the beginning A and B should paint the four squares containing the centre of the board. Then after each move of C the other children paint the rectangles obtained from the rectangle painted by C when it is rotated around the board centre by 90°, 180° and 270°. It is easy to see that a rectangle that does not contain the centre of the board can not overlap with its images under these rotations. Thus if C can move, so do D, A, and B.

Second solution. We call a *position* in our game a 1000×1000 board with a (probably empty) set of painted squares. Suppose that only two players play the game. We call a position *winning* if a player moving in this position can win irrespective of the opponent's play;we call a position *losing* if a player can not move in this position or every his move leads to a winning position (that is, his opponent can win irrespective of this move). We can prove by induction in the number of unpainted square that each position is either winning or losing one. The base is easily checked (all the positions with 0 or 1 unpainted squares are losing positions). Suppose all the positions with less than k unpainted squares are determined to be winning or losing. Consider an arbitrary position with k unpainted squares. All the moves in this position lead to positions with smaller number of unpainted squares. If all these positions are winning (or it is impossible to move) then our position is losing. If one can move from it to at least one losing position, then our position is winning.

Thus we learn whether the initial position (the position with no painted squares) is winning or losing. If it is winning, A and C can cooperate to avoid defeat: they simply should move from a winning position to a losing one, while B and C will be forced to move to winning positions. If the initial position is losing, similar cooperation is possible for B and D.

Let us suppose, for the sake of definiteness, that the initial position is winning (the argument for a losing position is similar). We shall prove that A and C can cooperate with B or D. For every winning position we fix one move to a losing position for A and C to make. In this way the moves of A and C are fixed, and the game is reduced to a game between B and D moving from one losing position to another. We can consider this new game; its positions are all the losing positions of the previous game. Since every move still leads to a position with less unpainted squares, in this game every position can be determined as a winning or a losing one. Then, if B starts in a winning position (considered as a position in the new game), then A, B, and C can cooperate against D: A and C make their fixed moves, and B always moves to a losing position of the new game. If B starts in a losing position, then A, C, and D can cooperate.