REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2019" (mathematics)

First day

Yakutsk 2019

The booklet contains the problems of XXVI International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical commission of Russian mathematical olympiad. The booklet was compiled by M. A. Antipov, A. S. Golovanov, K. P. Kokhas, A. S. Kuznetsov, N. Yu. Vlasova. Computer typesetting: K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

1. In a sequence a_1, a_2, $\ldots$ of real numbers the product a_1a_2 is negative, and to define a_n for $n > 2$ one pair (i, j) is chosen among all the pairs (i, j), $1 \leqslant i < j < n$, not chosen before, so that $a_i + a_j$ has minimum absolute value, and then a_n is set equal to $a_i + a_j$. Prove that $|a_i| < 1$ for some i.

(*A. Golovanov*)

2. A trapezoid $ABCD$ with $BC \parallel AD$ is given. The points B' and C' are symmetrical to B and C with respect to CD and AB, respectively. Prove that the midpoint of the segment joining the circumcentres of ABC' and $B'CD$ is equidistant from A and D.

(*A. Kuznetsov*)

3. The plan of a picture gallery is a chequered figure where each square is a room, and every room can be reached from each other by moving to adjacent rooms. A custodian in a room can watch all the rooms that can be reached from this room by one move of a chess queen (without leaving the gallery). What minimum number of custodians is sufficient to watch all the rooms in every gallery of n rooms ($n > 2$)?

(*H. Alpert, É. Roldán*)

4. A calculator can square a number or add 1 to it. It cannot add 1 two times in a row. By several operations it transformed a number x into a number $S > x^n + 1$ (x, n, S are positive integers). Prove that $S \geqslant x^n + x - 1$.

(*M. Antipov*)

Junior League

1. In a sequence a_1, a_2, $\ldots$ of integers the product a_1a_2 is negative, and to define a_n for $n > 2$ one pair (i, j) is chosen among all the pairs (i, j), $1 \leqslant i < j < n$, not chosen before, so that $a_i + a_j$ has minimum absolute value, and then a_n is set equal to $a_i + a_j$. Prove that $|a_i| < 1$ for some i.

(*A. Golovanov*)

2. A triangle ABC with $AB < AC$ is inscribed in a circle ω. Circles γ_1 and γ_2 touch the lines AB and AC, and their centres lie on the circumference of ω. Prove that C lies on a common external tangent to γ_1 and γ_2.

(*A. Kuznetsov*)

3. The plan of a picture gallery is a chequered figure where each square is a room, and every room can be reached from each other by moving to rooms adjacent by side. A custodian in a room can watch all the rooms that can be reached from this room by one move of a chess rook (without leaving the gallery). What minimum number of custodians is sufficient to watch all the rooms in every gallery of n rooms ($n > 1$)?

(*H. Alpert, É. Roldán*)

4. A quota of diplomas at the All-Russian Olympiad should be strictly less than 45 %. More than 20 students took part in the olympiad. After the olympiad the Authorities declared the results low because the quota of diplomas was significantly less than 45 %. The Jury responded that the quota was already maximum possible on this olympiad or any other olympiad with smaller number of participants. Then the Authorities ordered to increase the number of participants for the next olympiad so that the quota of diplomas became at least two times closer to 45 %. Prove that the number of participants should be at least doubled.

(*A. Golovanov*)

SOLUTIONS

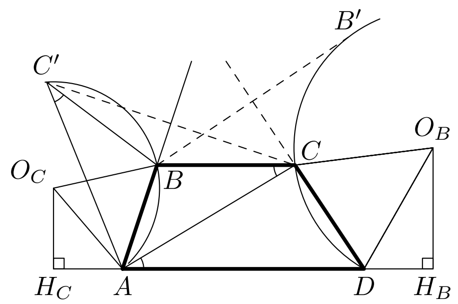
Senior League

1. Suppose the contrary: $|a_i| \geqslant 1$ for all i. Let M be the minimum positive term and m the maximum negative among the numbers $a_1, \ldots, a_n$. We shall prove that a number between $m+1$ and $M-1$ will appear in the sequence. Without loss of generality we may assume $M+m>0$. Since the sum $M+m$ has not been chosen, $|a_{n+1}| \leqslant |M+m|$. If a_{n+1} does not belong to the desired interval, it must lie between $m+1$ and $-(M+m)$.

Can it be that all the subsequent terms lie in the segment $\Delta = [-(M+m), m+1]$? If $a_i \in \Delta$ is obtained as a sum of two terms with different signs, the absolute values of these terms must be at least M. But the number of such terms is finite, and new large terms cannot appear. Therefore all the terms starting with some a_j are sums of negative terms and can be represented as sums of several numbers $a_1, \ldots, a_j$. Choosing integer s so that $M+m < s|m+1|$, we see that every such sum contains less than s terms. However, the number of such sums is finite.

Thus the distance between the minimum negative term and the maximum positive term will eventually decrease at least by 1. But this, too, cannot go on indefinitely, so the term with absolute value not exceeding 1 must appear in the sequence.

2. Let O_C and O_B be the circumcentres of ABC' and DCB' respectively. It is enough to prove that when the segment O_CO_B is projected onto the line AD, the midpoint of O_CO_B goes to the midpoint of the segment AD. Let H_C and H_B be the projections of O_C and O_B, respectively.



Thus the midpoint of O_CO_B is projected to the midpoint of H_CH_B, that is, we should prove that the midpoints of H_CH_B and AD coincide.

Instead we shall prove that $\overrightarrow{AH_C} = \overrightarrow{H_BD}$.

Since these vectors are collinear, it suffices to check that they have the same length and direction, i.e.,

$$AO_C \cos(\angle O_CAD) = DO_B \cos(\angle O_BDA).$$

To find the left-hand side, we apply the Law of Sines to the triangle ABC': $AO_C = \frac{AB}{2\sin(\angle AC'B)} = \frac{AB}{2\sin(\angle ACB)}$. Furthermore,

$$\angle O_C AD = \angle O_C AB + \angle BAD = 90^\circ - \angle AC'B + \angle BAD =$$
$$= 90^\circ - \angle ACB + \angle BAC + \angle CAD = 90^\circ + \angle BAC,$$

therefore, $\cos(\angle O_C AD) = \sin(\angle BAC)$. Finally,

$$AO_C \cos(\angle O_C AD) = \frac{AB}{2\sin(\angle ACB)} \cdot \sin(\angle BAC) = \frac{BC}{2},$$

where the last equality is the Law of Sines to the triangle ABC. Similarly, $DO_B \cos(\angle O_B DA) = \frac{BC}{2}$, and the equality is proved.

3. Answer: $\lfloor \frac{n}{3} \rfloor$ queens.

First we show that every polyomino with n tiles can be guarded by $\lfloor \frac{n}{3} \rfloor$ queens. We select one tile of the polyomino to be the root, and label all the tiles according to their distance from the root (where the distance between two tiles is the minimum number of steps needed to get from one to the other, such that each step goes from one tile of the polyomino to an adjacent tile of the polyomino). If possible, we select as the root a tile that is adjacent to only one other tile. If every tile of the polyomino is adjacent to more than one other tile, then we may select any tile as the root.

The tiles are partitioned into three sets according to whether their distance from the root is 0, 1, or 2 mod 3. If the 2 mod 3 set is empty, then a queen placed on the root guards the whole polyomino. Otherwise, we place queens on all the tiles in the smallest of the three sets. We claim that every tile is within distance 2 of at least one queen. If the queen set is the 0 mod 3 set, then from an arbitrary tile, we can find a queen within two steps by walking along a shortest path to the root. Similarly, if the queen set is the 1 mod 3 set, then from every tile except the root, we can find a queen within two steps by walking along a shortest path to the root, and we know that the root is also adjacent to a queen. If the queen set is the 2 mod 3 set, then from every tile of distance at least 2 from the root, we can find a queen within two steps toward the root, so it remains to check the root and the tiles adjacent to the root. We know that the root is distance exactly 2 from a queen. If it is adjacent to only one tile, then that tile is adjacent to a queen. Otherwise, every tile in the polyomino is adjacent to at least two tiles. Given a tile adjacent to the root, it must also be adjacent to another tile, and that tile is of distance 2 from the root and thus has a queen — for parity reasons, no two tiles adjacent to the root are adjacent to each other. Thus in all cases, every tile is within distance 2 of a queen, and thus, obviously, it is guarded by a queen.

We exhibit a polyomino with n tiles that needs $\lfloor \frac{n}{3} \rfloor$ queen guards. The construction is shown in Figure 2. If $n = 3m$, we make m rows of 3 tiles each, and stack them so that the center column contains the right-most tile of the first row, the left-most tile of the second, and the right-most tile of the third, and continues to alternate. Then if $n = 3m + 1$ or $n = 3m + 2$, we add the remaining one or two tiles to the bottom of the center column. Then no two of the m tiles furthest to the left and right can be guarded by the same queen, so at least $\lfloor \frac{n}{3} \rfloor$ queens are needed to guard this polyomino.

4. We consider all the numbers modulo $x^2 + x + 1$. If the number in the calculator is congruent to x, then the number at the next step is congruent to x^2 or $x+1 \equiv -x^2$. The number congruent to x^2 transforms either to $x^4 \equiv x$ or to $x^2 + 1 \equiv -x$. The numbers congruent to $-x^2$ and $-x$ are obtained by adding 1, so they can be only squared; this transforms them into numbers congruent to x and x^2, respectively. Thus we always get numbers congruent to $\pm x$ or $\pm x^2$ modulo $x^2 + x + 1$, or, equivalently, congruent to x, $x + 1$, x^2, $x^2 + 1$ modulo $x^2 + x + 1$.

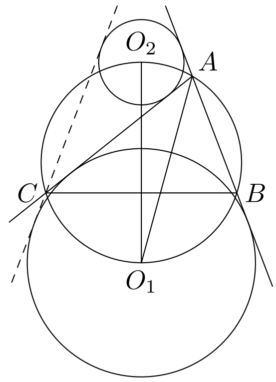
Note that if n gives the remainder r when divided by 3, then $x^n + 1 \equiv x^r + 1 \pmod{x^3 - 1}$ and therefore

$$x^n + 1 \equiv x^r + 1 \pmod{x^2 + x + 1}.$$

Thus $x^n + 1$ can leave only remainders 2, $x + 1$ or $x^2 + 1$ when divided by $x^2 + x + 1$. On the other hand, the numbers in the calculator can leave only remainders x, $x + 1$, x^2 or $x^2 + 1$. It is easy to check that if S (which gives one of the remainders x, $x + 1$, x^2, $x^2 + 1$) is greater than $x^n + 1$ (which gives one of the remainders 2, $x + 1$, $x^2 + 1$), then the difference is at least $x - 2$. It follows that if $S > x^n + 1$, then $S \geqslant x^n + 1 + (x - 2) = x^n + x - 1$.

Junior League

1. See problem 1, Senior league.



2. Let O_1, O_2 be the circumcentres of γ_1, γ_2. Note that the points O_1 and O_2 are at equal distances from the lines AB and AC, therefore, they lie on the bisectors of angle ABC (internal and external). Hence, O_1 and O_2 are intersection points of bisectors of angle ABC and ω, that is, O_1 and O_2 are midpoints of two arcs BC. It follows that B and C are symmetrical with respect to the line O_1O_2 passing through the centres of γ_1 and γ_2. Thus, if point B lies on a common tangent to these circles, so does C.

3. Answer: $\lfloor \frac{n}{2} \rfloor$ rooks.

First we show that every polyomino with n tiles can be guarded by $\lfloor \frac{n}{2} \rfloor$ rooks. We 2–color the polyomino according to the parity of the sum of coordinates of each tile; that is, tiles that share a side get opposite colors. We take the color with the smaller number of tiles, and place rooks on all tiles of that color, so that there are at most $\lfloor \frac{n}{2} \rfloor$ rooks. Because the interior of the polyomino is connected, every tile shares a side with some other tile. Thus every tile is guarded by at least one of the rooks.

Next we exhibit, for every positive integer n, a polyomino with n tiles such that the minimum number of rooks needed to guard it is $\lfloor \frac{n}{2} \rfloor$.

The construction, shown in Figure 1 for $n = 10$ and $n = 11$ tiles, consists of one center column with individual tiles attached on either side, alternating between left and right. This construction generates a polyomino for even $n = 2m$. For an odd number $n = 2m+1$ we construct a polyomino by placing a tile on the bottom-most part of center column of polyomino with $n-1$ tiles. To prove that $\lfloor \frac{n}{2} \rfloor$ rook guards are needed, we observe that there are exactly $\lfloor \frac{n}{2} \rfloor$ tiles not in the center column, and that no two of these can be guarded by the same rook.

4. Let q be the number of participants and the number of diplomas on the present olympiad respectively. It is known that $\frac{p}{q} < \frac{9}{20}$ and no fractions with denominator less than q lie between $\frac{p}{q}$ and $\frac{9}{20}$. In all future olympiads the number of participants n and the number of diplomas m must satisfy $\frac{p}{q} < \frac{m}{n} < \frac{9}{20}$, $\frac{9}{20} - \frac{p}{q} \geqslant 2\left(\frac{9}{20} - \frac{m}{n}\right)$.

First we prove the following general proposition.

Lemma. Let p, q, r, $s < q$ be positive integers such that the fractions $\frac{p}{q}$ and $\frac{r}{s}$ are irreducible and no fraction with denominator not exceeding

q lies between these fractions. Then for every fraction $\frac{m}{n}$ between $\frac{p}{q}$ and $\frac{r}{s}$ there are positive integers x and y such that $m = px + ry$, $n = qx + sy$.

Proof. The system $m = px + ry$, $n = qx + sy$ admits the only solution $x = \frac{ms-nr}{ps-qr}$, $y = \frac{pn-qm}{ps-qr}$. These x and y are rational and positive; this becomes evident when the latter equations are re-written as

$$x = \frac{n}{q} \cdot \frac{\frac{m}{n} - \frac{r}{s}}{\frac{p}{q} - \frac{r}{s}}, \qquad y = \frac{n}{s} \cdot \frac{\frac{p}{q} - \frac{m}{n}}{\frac{p}{q} - \frac{r}{s}}. \qquad (*)$$

It remains to prove that x and y are integral. Suppose the contrary: at least one of the numbers $\alpha = \{x\}$, $\beta = \{y\}$ is not 0. Then the numbers $m' = p\alpha + r\beta$, $n' = q\alpha + s\beta$, $m'' = p(1-\alpha) + r(1-\beta)$, $n'' = q(1-\alpha) + s(1-\beta)$ are positive integers and the fractions $\frac{m'}{n'}$, $\frac{m''}{n''}$ lie between $\frac{p}{q}$ and $\frac{r}{s}$. On the other hand, $n' + n'' = q + s < 2q$, that is, at least one of the numbers n' and n'' is less than q, a contradiction.

To solve our problem, we apply the lemma to the fractions $\frac{p}{q}$ and $\frac{9}{20}$, that is, take $r = 9$, $s = 20$. Then the lemma gives us $m = px + 9y$, $n = qx + 20y$ with positive integral x and y; in particular, $x \geqslant 1$. In the first equation (*) the second factor $\frac{\frac{m}{n} - \frac{r}{s}}{\frac{p}{q} - \frac{r}{s}}$ does not exceed $\frac{1}{2}$, therefore, the first factor $\frac{n}{q}$ is at least 2, q.e.d.