REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2018" (mathematics)

Second day

Yakutsk 2018

The booklet contains the problems of XXV International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by A. V. Antropov, S. L. Berlov, A. S. Golovanov, K. P. Kokhas, A. S. Kuznetsov, F. V. Petrov, N. Y. Vlasova. Computer typesetting: K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

5. A prime p and a positive integer n are given. The product

$$(1^3+1)(2^3+1)\dots((n-1)^3+1)(n^3+1)$$

is divisible by p^3. Prove that $p \leqslant n+1$. (*Z. Luria*)

6. Prove the inequality

$$(x^3+2y^2+3z)(4y^3+5z^2+6x)(7z^3+8x^2+9y) \geqslant 720(xy+yz+xz)$$

for x, y, $z \geqslant 1$. (*K. Kokhas*)

7. A school has three senior classes of M students each. Every student knows at least $\frac{3}{4}M$ people in each of the two other classes. Prove that the school can send M non-intersecting teams to the olympiad so that each team consists of 3 students from different classes who know each other. (*C. Magyar, R. Martin*)

8. Quadrilateral $ABCD$ with perpendicular diagonals is inscribed in a circle with centre O. The tangents to this circle at A and C together with line BD form the triangle Δ. Prove that the circumcircles of BOD and Δ are tangent. (*A. Kuznetsov*)

Junior League

5. 99 identical balls lie on a table. 50 balls are made of copper, and 49 balls are made of zinc. The assistant numbered the balls. One spectrometer test is applied to 2 balls and allows to determine whether they are made of the same metal or not. However, the results of the test can be obtained only the next day. What minimum number of tests is required to determine the material of each ball if all the tests should be performed today? (*N. Vlasova, S. Berlov*)

6. The numbers 1, 2, 3, …, 1024 are written on a blackboard. They are divided into pairs. Then each pair is wiped off the board and non-negative difference of its numbers is written on the board instead. 512 numbers obtained in this way are divided into pairs and so on. One number remains on the blackboard after ten such operations. Determine all its possible values. (*A. Golovanov*)

7. Prove the inequality

$$(x^3+2y^2+3z)(4y^3+5z^2+6x)(7z^3+8x^2+9y) \geqslant 720(xy+yz+xz)$$

for x, y, $z \geqslant 1$.

(*K. Kokhas*)

8. Quadrilateral $ABCD$ with perpendicular diagonals is inscribed in a circle with centre O. The tangents to this circle at A and C together with line BD form the triangle Δ. Let ω be the circumcircle of OAC. Prove that the circumcircles of BOD and Δ are tangent and their common point belongs to ω. (*A. Kuznetsov*)

SOLUTIONS

Senior League

5. Suppose that $p > n+1$. Let p divide k^3+1 for some k, $1 \leqslant k \leqslant n$. Since $k^3+1 = (k+1)(k^2-k+1)$ and $k+1 < p$, p must divide k^2-k+1. The number k^2-k+1 is less than p^2 and therefore cannot be divisible by p^2.

Thus there are three different positive numbers k, m, ℓ not exceeding n such that p divides k^2-k+1, m^2-m+1, $\ell^2-\ell+1$.

Then p divides

$$(k^2-k+1)-(m^2-m+1) = k^2-m^2-(k-m) = (k-m)(k+m+1).$$

Note that $0 < |k-m| < n < p$, so p divides $k+m+1$. But $0 < k+m+1 < 2n+1 < 2p$, that is, $k+m+1 = p$. Similarly, $k+\ell+1 = p$. Therefore $m = \ell$, a contradiction.

6. Expanding the LHS we obtain

$$\begin{aligned}
&(x^3+2y^2+3z)(4y^3+5z^2+6x)(7z^3+8x^2+9y) = \\
&= (48x^6+72y^6+105z^6)+ && (1)\\
&+ (54x^4y+32x^5y^3+96x^3y^2+64x^2y^5+108xy^3+36x^3y^4)+ && (2)\\
&+ (42x^3z^3+40x^5z^2+144x^3z+35x^3z^5+126xz^4+120x^2z^3)+ && (3)\\
&+ (56y^5z^3+90y^3z^2+108y^4z+135z^3y+84z^4y^3+70y^2z^5)+ && (4)\\
&+ (28x^3y^3z^3+45x^3yz^2+84xy^2z^3+80x^2y^2z^2+162xyz+96x^2y^3z) && (5)
\end{aligned}$$

Reducing all the exponents to 1 we get a lower estimate of the expressions in the second, third, and fourth lines. To estimate the first line we use obvious inequality

$$\begin{aligned}
48x^6+72y^6+105z^6 &= 7.5(x^6+y^6)+64.5(y^6+z^6)+40.5(x^6+z^6) \geqslant \\
&\geqslant 15x^3y^3+129y^3z^3+81x^3z^3 \geqslant 15xy+129yz+81xz.
\end{aligned}$$

Note that the sums of coefficients on the left and on the right are equal: $48+72+105 = 15+129+81$. Thus the expression in the first four lines is greater or equal to

$$405xy+588xz+672yz.$$

It remains to check that the expression in the fifth line is not less than

$$315xy + 132xz + 48yz,$$

a task presenting no difficulties.

7. Consider the graph G where vertices are our students, and the edge between two vertice means that two students are in different classes and know each other. Since we do not consider the edges between the students of the same class, this graph is tripartite (each part corresponds to one of the classes).

First we choose two parts and prove that there is a perfect matching in the subgraph made of these two parts.

L e m m a. If in a bipartite graph with M vertices in each part the degree of each vertex is at least $M/2$, the graph has a perfect matching.

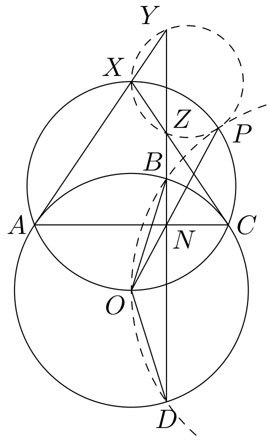
P r o o f. We check the condition of Hall's marriage theorem: every k people in the first part know at least k people in the second part.

If $k \leqslant \frac{M}{2}$ the condition is satisfied since any one of the k people knows at least $\frac{M}{2} \geqslant k$ people in the second part. If $k > \frac{M}{2}$ then every man in the second part knows at least one of these k people, therefore any k people in the first part know M people in the second part. The lemma is proved.

Thus graph G admits a perfect matching between the vertices of the second part and the vertices of the third part. Now we consider another bipartite graph H: its first part coincides with the first part of G, and the vertices of the second part are edges of the perfect matching. In H, we connect the vertex of the first part to an edge in the matching when both ends of this edge are connected to this vertex in G. The degree of each vertex in H is at least $\frac{M}{2}$. Indeed, each end of an edge is connected in G to at least $\frac{3}{4}M$ vertices of the first part; hence there are at least $\frac{M}{2}$ vertices connected to both ends. On the other hand, each vertex V of the first part in G is connected to at least $\frac{3}{4}M$ vertices of the second part and at least $\frac{3}{4}M$ vertices of the third part. Thus there are at least $\frac{M}{2}$ edges of the matching such that V is connected to both their ends. The lemma implies that H also admit a perfect matching. Now the desired triples can be formed of vertices of the first part and the ends of matching edges.

8. Let X be the common point of the tangents at A and C, Y the common point of the lines AX and BD, Z the common point of the lines CX and BD, N the common point of the lines AC and BD. (Thus the triangle Δ is simply XYZ).

Since AX and CX are tangents, $\angle OAX = \angle OCX = 90^\circ$, and $OAXC$ is cyclic.



Lemma. The circumcircles of OBD, Δ and $OAXC$ have a common point.

Proof. Note that BD is the radical axis of the circumcircles of OBD and $ABCD$, and AC is the radical axis of the circumcircles of $ABCD$ $OAXC$. Therefore N is the radical centre of the circles OBD, $ABCD$, $OAXC$, and the line ON goes through the common point $P \neq O$ of the circumcircles of OBD and $OAXC$.

To prove the lemma we need to prove that the quadrilateral $ZXYP$ is cyclic.

Let $\angle XAC = \angle XCA = \alpha$, $\angle PAC = \beta$. Since $YZ \parallel XO$, $\angle XYZ = \angle AXO = 90^\circ - \alpha$. It is enough to prove that $\angle XPZ = 90^\circ - \alpha$.

Choose Z' on the segment XC so that $\angle XPZ' = 90^\circ - \alpha$. It is easy to see that $\angle OPC = \angle OXC = 90^\circ - \alpha$. We have also $\angle XPO = 90^\circ$, therefore $\angle Z'PN = \alpha = \angle Z'CN$ and the quadrilateral $Z'PCN$ is cyclic. Hence $\angle Z'NC = 180^\circ - \angle Z'PC = 180^\circ - \alpha - (90^\circ - \alpha) = 90^\circ$, that is, $Z = Z'$ and $XYPZ$ is cyclic. The lemma is proved.

Now we note that the diameter of the circumcircle of $OAXC$ touches the circumcircle of $XYPZ$, therefore the tangents at X to these circles are perpendicular. Then their tangents at P must be perpendicular too. Similarly, the tangents to the circumcircles of $OAXC$ and OBD at P are perpendicular. Thus the tangents at P to the circumcircles of $XYPZ$ and OBD coincide, and these circumcircle are tangent.

Junior League

5. The answer is 98.

An algorithm of 98 tests is the following. We choose one ball A and compare it with each of the rest. When the results are out the balls are divided into two groups: the balls made of the same metal as A (including A), and the balls made of another metal. One of the groups contains 50 balls, they are made of copper, and the rest is zinc.

To prove the estimate, suppose we made an algorithm with at most 97 tests. Consider a graph where vertices are balls and an edge means that the corresonding pair of balls was tested together. Since the number of edges is less than 98, the graph is unconnected.

If the graph contains a component with even number of vertices, we can color 50 vertices white and 49 black so that the even component contains equal number of black and white vertices. If white vertices are copper balls and black vertices are zinc balls then a test corresponding to an edge determines whether the ends of this edge have the same colour. Obviously changing the colours of all vertices in the even component does not affect the results of the tests.

If there are no even components, the graph contains at least three odd components. Then we can do the same trick with the union of vertices of two components.

6. Answer: it can be any even number between 0 and 1022.

To obtain a number $2k$, $0 \leqslant k \leqslant 511$, we form one pair of the numbers 1 and $2k + 2$, and divide all other numbers into pair of consecutive integers. Then the first operation gives us the number $2k + 1$ and 511 ones. In the second step we get $2k$ and 255 zeroes. Obviously the last number will be $2k$.

Clearly we can not obtain 1024. Moreover, the operation preserves the parity of the sum of all numbers. Initially this sum equals $1 + 2 + \ldots + 1024 = 1025 \cdot 512$, an even number, so the last number must be even too.

7. See Problem 6, senior league.

8. See Problem 8, senior league.