REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD
”TUYMAADA-2018”
(mathematics)

First day

Yakutsk 2018

The booklet contains the problems of XXV International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by

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Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

1. Do there exist three different quadratic trinomials $f(x)$, $g(x)$, $h(x)$ such that the roots of the equation $f(x) = g(x)$ are 1 and 4, the roots of the equation $g(x) = h(x)$ are 2 and 5, and the roots of the equation $h(x) = f(x)$ are 3 and 6?

(A. Golovanov)

2. 2550 rooks and k pawns are arranged on a 100×100 board. The rooks cannot leap over pawns. For which minimum k it is possible that no rook can capture any other rook?

(N. Vlasova)

3. A point P on the side AB of a triangle ABC and points S and T on the sides AC and BC are such that $AP = AS$ and $BP = BT$. The circumcircle of PST meets the sides AB and BC again at Q and R, respectively. The lines PS and QR meet at L. Prove that the line CL bisects the segment PQ.

(A. Antropov)

4. Prove that for every positive integers $d > 1$ and m the sequence $a_n = 2^{2^n} + d$ contains two terms a_k and a_ℓ $(k \neq \ell)$ such that their greatest common divisor is greater than m.

(T. Hakobyan)

Junior League

1. Real numbers $a \neq 0$, b, c are given. Prove that there is a polynomial $P(x)$ with real coefficients such that the polynomial $x^2 + 1$ divides the polynomial $aP^2(x) + bP(x) + c$.

(A. Golovanov)

2. A circle touches the side AB of the triangle ABC at A, touches the side BC at P and intersects the side AC at Q. The line symmetrical to PQ with respect to AC meets the line AP at X. Prove that $PC = CX$.

(S. Berlov)

3. 2551 rooks and k pawns are arranged on a 100×100 board. The rooks cannot leap over pawns. For which minimum k it is possible that no rook can capture any other rook?

(A. Kuznetsov)

4. Prove that for every odd positive integer $d > 1$ and every positive integer m the sequence $a_n = 2^{2^n} + d$ contains two terms a_k and a_ℓ $(k \neq \ell)$ such that their greatest common divisor is greater than m.

(T. Hakobyan)

SOLUTIONS

Senior League

1. The statement on the roots of equation $f(x) = g(x)$ means that the difference $f(x) - g(x)$ is of the form $a(x-1)(x-4)$ (since it is a polynomial of degree not exceeding 2 with roots 1 and 4). Applying the same argument to the other differences we can write the equality $(f(x) - g(x)) + (g(x) - h(x)) = (f(x) - h(x))$ as

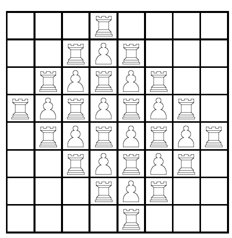
$$a(x-1)(x-4) + b(x-2)(x-5) = c(x-3)(x-6).$$

Putting $x = 1$ we get $4b = 10c$; putting $x = 4$ we get $-2b = -2c$. This is possible only when $b = c = 0$, but then f, g, h coincide, a contradiction.

2. Answer: for $k = 2450$.

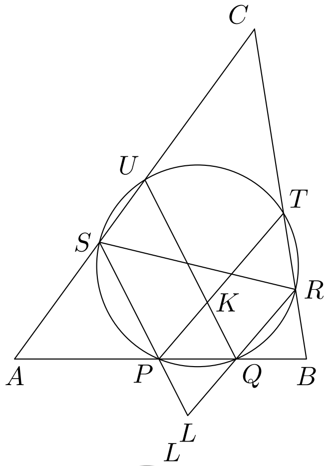
To prove the estimate we start in each rook a downward segment ending at the first pawn it meets or, if no such pawn is found, at the lower border of the table. In every row there is at most one segment ending at the border. At most one segment ends at each pawn. Thus the number of pawns is at least 2450.

To construct an example, we start with putting a rook in the fiftieth square of the upper row. In the second row we put a pawn in the fiftieth square and two rooks to the left and to the right of it. Then all the rows down to the 50th are filled in the following way: we put a rook under each pawn of the preceding row and a pawn under each rook, and add one rook to the left of the leftmost pawn and one rook to the right of the rightmost pawn.

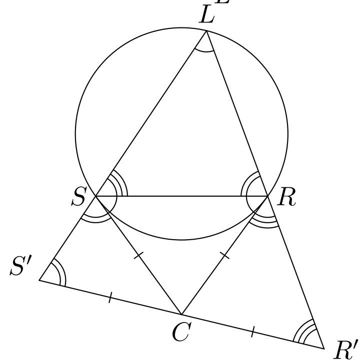


The lower half of the board is symmetrical to the upper one with respect to the center of the board (in the picture 8×8 board is shown).

In this arrangement rooks and pawn alternate in each row and column. This is obvious for rows; in columns we should only look at the 50th and 51th rows. Note that the 50th row contains rooks on all the odd positions and pawns on all the even positions except the last one. After the reflection the 51th row will contain rooks on all the even positions and pawns on all the odd positions except the first one; thus the neighbouring figures of these two rows are different, and our arrangment satisfies the condition. Obviously it contains 2550 rooks and 2450 pawns.



3. Let the circumcircle of PST meets AC again at U, and K is the common point of PT and QU. Applying Pascal's theorem to $RTPSUQ$ we see that K, L, C are collinear. Since $AP = AS$, $PQUS$ is an isosceles trapezoid and $PS \parallel QU$, similarly, $QR \parallel PT$. It remains to note that $QLPK$ is a parallelogram and its diagonal KL bisects its other diagonal PQ.



Second solution.

Lemma. Let the angles of triangle SLR satisfy $\angle LSR < 90^\circ$, $\angle SRL < 90^\circ$ and the tangents to the circumcircle of SLR at S and R meet at C. Then LC is the symmedian of triangle SLR.

Proof. Note that $\angle RCS = \angle SRC = \angle SLR$. Therefore the angles $\angle CSL = 180^\circ - \angle SRL$ and $\angle CRL = 180^\circ - \angle RSL$ are obtuse. We take ponts S' and R' on the rays LS and LR respectively so that $CS' = CS = CR = CR'$. These points lie on the extended sides. Then $\angle CSS' = \angle CS'S = \angle SRL$, hence $\angle SCS' = 180^\circ - 2\angle SRL$. Similarly, $\angle RCR' = 180^\circ - 2\angle RSL$. Therefore

$$\angle S'CR' = 180^\circ - 2\angle SRL + 180^\circ - 2\angle SLR + 180^\circ - 2\angle RSL = 180^\circ,$$

i. e. LC is the median of $S'LR$. It is easy to see that triangles LSR and $LR'S'$ are similar, therefore the ray LC goes along the median of $\triangle PQL$. The lemma is proved.

The solution of the problem follows almost immediately.

Obviously

$$\angle SLR = \angle RSC = \angle SCR = \frac{1}{2}(\angle BAC + \angle ABC).$$

Then SC and RC are tangent to the circumcircle of SLR. It follows from the lemma that LC is symmedian in $\triangle SRL$. The line PQ is antiparallel to SR, therefore, the ray LC goes along the median of $\triangle PQL$.

4. Let $v_p(n)$ denote the maximum k such that a positive integer n is divisible by p^k, where p is a prime.

Suppose there exist m and d such that $(a_k, a_\ell) \leqslant m$ for all k and ℓ. It follows that if p^t divides a_k and a_ℓ for some positive integers t, k, ℓ

$(k \neq \ell)$ and prime p then $p^t < m$. In other words, for every prime p the sequence $\{v_p(a_k)\}$ is bounded.

L e m m a 1. If positive integers n and k satisfy the inequality $v_2(k) < n$ then k divides $2^\ell - 2^n$ for some $\ell > n$.

P r o o f. Let $k = 2^a b$, where b is odd. We have $a < n$. There exists m such that b divides $2^m - 1$. Then $2^a b$ divides $2^{n+m} - 2^n = 2^n(2^m - 1)$.

L e m m a 2. If p divides a_n for some prime $p > m$ and integral n then $p \equiv 1 \pmod{2^n}$.

P r o o f. Suppose 2^n does not divide $p - 1$. It follows from Lemma 1 that $p - 1$ divides $2^\ell - 2^n$ for some $\ell > n$. Then

$$a_\ell - a_n = 2^{2^\ell} - 2^{2^n} = 2^{2^n}(2^{2^\ell - 2^n} - 1).$$

The difference $2^{2^\ell - 2^n} - 1$ is divisible by p because $2^\ell - 2^n$ is divisible by $p - 1$. Therefore $p \leqslant (a_n, a_\ell) \leqslant m$, a contradiction.

L e m m a 3. d is a power of 2.

P r o o f. For each n we write $a_n = 2^{k_n} b_n c_n$, where b_n contains only odd prime divisors of a_n that are less than m and c_n contains those that are not less than m (b_n or c_n can be equal to 1).

It follows from Lemma 2 that $c_n \equiv 1 \pmod{2^n}$, therefore $a_n \equiv d \equiv 2^{k_n} b_n \pmod{2^n}$. The number of primes less than m is finite and for each of them the sequence $\{v_p(a_n)\}$ is bounded. Moreover, $k_n = v_2(d)$ when $n > v_2(d)$. This means that there exists M such that $2^{k_n} b_n < M$ for every integer n, that is, $2^{k_n} b_n = d$ for sufficiently large n.

Thus d divides $a_n = 2^{2^n} + d$ for sufficiently large n, so it also divides 2^{2^n}, and d is a power of 2.

L e m m a 4. For large enough n there exists $\ell > n$ such that a_n divides a_ℓ.

P r o o f. Let $d = 2^k$. Choose $n > v_2(k)$; then $v_2(2^n - k) = v_2(k)$. It follows from Lemma 1 that there exists ℓ such that $2^n - k$ divides $2^\ell - 2^n$ and therefore $2^\ell - k$.

Since $v_2(2^\ell - k) = v_2(k) = v_2(2^n - k)$, the number $(2^\ell - k)/(2^n - k)$ is odd, thus $2^{2^n - k} + 1$ divides $2^{2^\ell - k} + 1$. Multiplying by 2^k we get that a_n divides a_ℓ.

It follows from Lemma 4 that $a_n = (a_n, a_\ell) \leqslant m$ for each $n > v_2(k)$, a contradiction.

Junior League

1. First solution. It is obviously enough that x^2+1 divides $P^2(x)+\frac{b}{a}P(x)+\frac{c}{a}$; therefore we may assume $a=1$. We prove that it is always possible to find a polynomial of the form $P(x)=rx+s$. For such polynomial

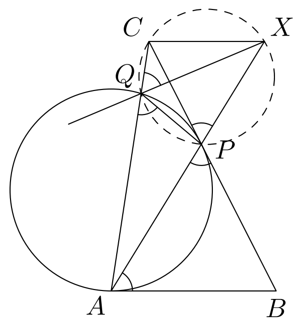
$$P^2(x)+bP(x)+c=(rx+s)^2+b(rx+s)+c=r^2x^2+(2rs+br)x+(s^2+bs+c),$$

and it suffices to satisfy the conditions $2rs+br=0$ and $r^2=s^2+bs+c$. If the equation $s^2+bs+c=0$ has a root s_0 (that is, if $b^2\geqslant 4c$), these conditions are satisfied by $r=0$ and $s=s_0$ (then $P(x)=s_0$ is constant). Otherwise, the first condition is satisfied by $s=-\frac{b}{2}$. Then the second condition becomes $r^2=c-\frac{b^2}{4}$ where RHS is positive, and the desired r exists. In this case we produce suspiciously familiar-looking polynomial $P(x)=-\frac{b}{2}\pm\frac{\sqrt{4c-b^2}}{2}x$.

The second solution is intended for those who alredy know what a complex number is, or for those who still do not know but badly want to. Clearly, if the polynomial at^2+bt+c has a real root, then this root (considered as a constant polynomial $P(x)$) satisfies the condition. If not, at^2+bt+c has two conjugate complex roots $r+si$ and $r-si$. Let us prove that $P(x)=r+sx$ satisfies the condition. Indeed, it is chosen so that it vanishes at $x=i$ and $x=-i$. Therefore it is divisible by $(x+i)(x-i)=x^2+1$.

What we really wanted to find in the first solution was, obviously, not the polynomial $P(x)$ but the remainder when it is divided by x^2+1 (since the required divisibility depends on the remainder only). In fact we operated in the arithmetic of residues modulo x^2+1, where residue x multiplied by itself is -1. It happened that every quadratic equation has a root in this arithmetic. This arithmetic is the arithetic of complex numbers, and every non-constant polynomial has a root in it.

2. Since BA and BP are tangents, we have $BA=BP$, and therefore $\angle BAP=\angle BPA$. Moreover, $\angle BPA=\angle XPC$ (these angles are vertical), $\angle BAP=\angle PQA$ (the angle between chord and tangent). By symmetry we have $\angle PQA=\angle XQC$, that is, $\angle XQC=\angle XPC$, and quadrilateral $PQCX$ is cyclic. Thus $\angle CXP=\angle PQA=\angle XPC$, and triangle PCX is isosceles.



3. Answer: for $k=2452$.

To prove the estimate we put a pawn under each square of the last row and start in each rook a downward segment ending at the first occupied square it enters (obviously this square contains a pawn). At most one segment ends at each pawn. Thus the number of pawns is at least 2551, and the board contains at least 2451 “real” pawns.

Suppose we arranged 2551 rooks and (exactly) 2551 pawns. Then each pawn is at the end of exactly one segment. We prove that in this case for each $k \leqslant 50$ the k-th row (counted from below) cannot contain more than k rooks.

Assume that the k-th row contains at least $k + 1$ rooks. Consider $k \times 100$ rectangle adjacent to the lower side of the board. In each row of the rectangle the number of rooks exceeds the number of pawns at most by 1, therefore the total number of rooks in the rectangle exceeds tha number of pawns at most by k. On the other hand, no column of this rectangle can contain more pawns than rooks, becuse every rook is the end of a segment. Besides, each of $k + 1$ columns containing the rooks of the k-th row has more rooks than pawns. Thus the number of rooks exceeds the number of pawns at least by $k + 1$, a contradiction.

Now 50×100 rectangle adjacent to the lower side of the board contains at most $1+2+\ldots+50 = 1275$ rooks. The same is true for the upper half of the board, and the total number of rooks is at most 2550, contrary to our supposition.

The example can be obtained by arranging 2550 rooks and 2450 pawns as in the problem 2, senior league, and adding one rook in a corner and two pawns in the neighbouring squares.

4. See Problem 4, senior league (the first three lemmas).