REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION

INTERNATIONAL OLYMPIAD "TUYMAADA-2016" (mathematics)

First day

Yakutsk 2016

The booklet contains the problems of XXII International school students olympiad "Tuymaada"in mathematics.

The problems were prepared with the participation of members of the Methodical comission of Russian mathematical olympiad. The booklet was compiled by S. L. Berlov, A. S. Chukhnov, V. L. Dolnikov A. S. Golovanov, L. A. Emelyanov, K. P. Kokhas, F. V. Petrov, A. I. Khabrov, Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

1. The sequence (a_n) is defined by $a_1 = 0$,

$$a_{n+1} = \frac{a_1 + a_2 + \ldots + a_n}{n} + 1.$$

Prove that $a_{2016} > \frac{1}{2} + a_{1000}$.

(A. Golovanov)

2. A cube stands on one of the squares of an infinite chessboard. On each face of the cube there is an arrow pointing in one of the four directions parallel to the sides of the face. Anton looks on the cube from above and rolls it over an edge in the direction pointed by the arrow on the top face. Prove that the cube cannot cover any 5×5 square.

(A.Chukhnov)

3. Altitudes AA_1, BB_1, CC_1 of an acute triangle ABC meet at H. A_0, B_0, C_0 are the midpoints of BC, CA, AB respectively. Points A_2, B_2, C_2 on the segments AH, BH, HC_1 respectively are such that $\angle A_0B_2A_2 = \angle B_0C_2B_2 = \angle C_0A_2C_2 = 90°$. Prove that the lines AC_2, BA_2, CB_2 are concurrent.

(A.Pastor)

4. For each positive integer k determine the number of solutions of the equation

$$8^k = x^3 + y^3 + z^3 - 3xyz$$

in non-negative integers x, y, z, $0\leqslant x\leqslant y\leqslant z$.

(V. Shevelev)

Junior league

1. Tanya and Serezha have a heap of 2016 candies. They make moves in turn, Tanya moves first. At each move a player can eat either one candy or (if the number of candies is even at the moment) exactly half of all candies. The player that cannot move loses. Which of the players has a winning strategy?

(A. Golovanov)

2. The point D on the altitude AA_1 of an acute triangle ABC is such that $\angle BDC = 90°$; H is the orthocentre of ABC. A circle with diameter AH is constructed. Prove that the tangent drawn from B to this circle is equal to BD.

(L.Emelyanov)

3. A cube stands on one of the squares of an infinite chessboard. On each face of the cube there is an arrow pointing in one of the four directions parallel to the sides of the face. Anton looks on the cube from above and rolls it over an edge in the direction pointed by the arrow on the top face. Prove that the cube cannot cover any 5×5 square.

(A.Chukhnov)

4. Non-negative numbers a, b, c satisfy $a^2 + b^2 + c^2 \geqslant 3$. Prove the inequality

$$(a + b + c)^3 \geqslant 9(ab + bc + ca).$$

(A.Khrabrov)

SOLUTIONS

Senior league

1. We write the given relation in the form

$$na_{n+1} = a_1 + \ldots + a_n + n.$$

Now we write the same formula for a_n:

$$(n-1)a_n = a_1 + \ldots + a_{n-1} + n - 1.$$

Subtracting these equations we get

$$a_{n+1} = a_n + \frac{1}{n}.$$

It follows that

$$a_{2016} = a_{1000} + \frac{1}{1000} + \frac{1}{1001} + \ldots + \frac{1}{2015}$$

and it remains to prove that $\frac{1}{1000} + \frac{1}{1001} + \ldots + \frac{1}{2015} > \frac{1}{2}$. This is a standard estimate: there are 2016 terms in the sum, and each of them is at least $\frac{1}{2015}$, therefore

$$\frac{1}{1000} + \frac{1}{1001} + \ldots + \frac{1}{2015} > 1016 \cdot \frac{1}{2015} > \frac{1}{2}.$$

2. Consider the first moment t_1 when some face of the cube, which has already been on the top (at some moment t_0), is on the top again. Denote the present position of its centre by B and its position at the moment t_0 by A. Between the moments t_0 and t_1 the cube has been rolled $m \leqslant 6$ times. There are three possible cases.

1) The arrow points in the same direction at these two moments.

Then every m rollings move the cube by vector $\overrightarrow{AB}$. The case $A = B$ is trivial. If the vector $\overrightarrow{AB}$ is horizontal, a 5×5 square cannot be covered because in at most 6 rollings the cube can not visit 5 neighbouring horizontal rows. The same argument can be applied to the case when $\overrightarrow{AB}$ is vertical. Finally we consider the case when $\overrightarrow{AB}$ is not parallel to the sides of board squares and the cube has some moment covered a 5×5 square.

It is informally obvious that the trajectory of the cube is to "narrow" to cover a 5×5 square. To avoid the study of particular trajectories we use asymptotic argument. If the trajectory is purely periodic then in m more moves the cube covers the image of the 5×5 square under translation by $\overrightarrow{AB}$. Since the new square contains at least 9 squares not covered by the old one, in $m(k-1)$ more moves the cube covers at least k large squares containing at least $9k+16$ unit squares. On the other hand, if the first 5×5 square has been covered in s moves, k large squares have been covered in $s+m(k-1)$ moves, that is, $s+m(k-1) \geqslant 9k+16$ for all positive integers k. This is possible only when $m \geqslant 9$, a contradiction.

If the trajectory is not purely periodic, the above argument only strengthens: if non-periodic part contains q squares, then the period does not exceed $6-q$, in the next $m \leqslant 6-q$ moves the cube covers the translation of 5×5 square without possibly q squares (corresponding to the non-periodic part). Therefore it adds at least $9-q$ new squares. Continuing the argument we get $m \geqslant 9-q$, again a contradiction.

2) The arrows at the moments t_0 and t_1 point in opposite directions.

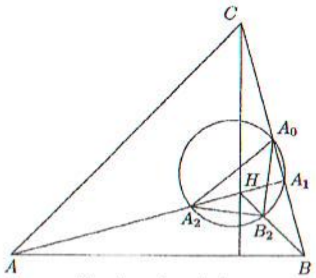
Then next m rollings move the cube back to A and its trajectory becomes periodic. The period covers at most 12 squares; adding at most 5 squares covered before t_0, we get at most 17 squares.

3) At t_1 the arrow is rotated by 90° with respect to its position at t_0.

Then next m rollings move the cube by the vector $\overrightarrow{AB}$ rotated by 90°, m more rollings move it by the vector $\overrightarrow{AB}$ rotated by 180° and then by the vector $\overrightarrow{AB}$, rotated by 270° to A. This period covers $4m \leqslant 24$ squares. Therefore if the trajectory is purely periodic, it covers, too, at most 24 squares

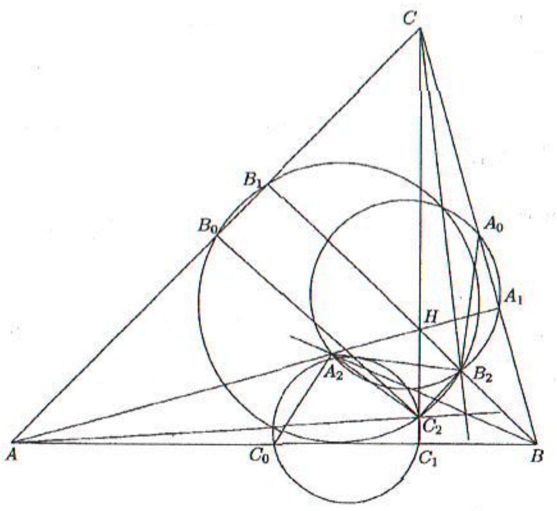
It the trajectory is not purely periodic and some faces were on the top before the moment t_0, then after that none of them can be the top face, and the number of covered squares is even less. Indeed, if k faces have been at the top before the moment t_0, then $m \leqslant 6-k$ and the total number of covered squares does not exceed $k + 4(6-k) < 24$.

3. Since $\angle A_0B_2A_2 = \angle A_2A_1A_0 = 90°$, the points B_2 and A_1 belong to the circle with diameter A_0A_2. The power of C with respect to that circle is $CA_0 \cdot CA_1 = \frac{1}{2}BC \cdot AC \cdot \cos C$.



Similarly, the power of C with respect to the circle with diameter B_0B_2 is $CB_0 \cdot CB_1 = \frac{1}{2}AC \cdot BC \cdot \cos C$.

Since these expressions are equal, CB_2 is the radical axis of these two circles. Then the lines CB_2, BA_2, AC_2 are radical axes of three pairs of circles and therefore meet at their radical centre.



4. The answer is $k+1$.

We prove it by induction in k.

Let $k=0$. Note that

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-xz).$$

We should solve the equation

$$1=(x+y+z)(x^2+y^2+z^2-xy-yz-xz).$$

The first factor is at least 1 and the other is a positive integer. Thus in this case $x=y=0$, $z=1$ is the only solution.

The induction step: if the numbers x, y, z giving a representation of 8^{k-1} are multiplied by 2, the resulting numbers give a representation of 8^k. It remains to prove that there is a unique representation of 8^k with an odd number among x, y, z. Note that in such representation the sum $x+y+z$ is greater than 1 and therefore even, that is, there are exactly two odd numbers among x, y, z. Then $x^2+y^2+z^2-xy-yz-xz$ is odd and thus equals 1. Now

$$1=x^2+y^2+z^2-xy-yz-xz=\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right),$$

so $z-x=1$ and y is equal to one of the numbers x and z. If $x=y=a$, $z=a+1$ we have

$$x^3+y^3+z^3-3xyz=3a+1,$$

and if $x=a-1$, $y=z=a$ then

$$x^3+y^3+z^3-3xyz=3a-1.$$

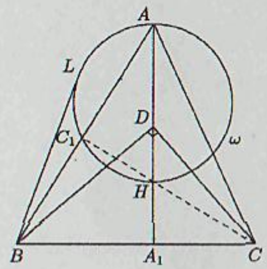
Therefore the desired representation is uniquely defined by the residue of 8^k modulo 3.

Младшая лига

1. Tanya can win.

She has a forced strategy, that is, a sttategy leaving no initiative to the opposition. She ought to take 1 candy almost to the end of the game. Then after her move the heap always contains an odd number of candies, and Serezha's move is forced: he, too, must take one candy. These events diminish the heap by 2 candies. When only 4 candies remain in the heap, Tanya makes an extraordinary move and takes half the heap, leaving 2 candies for Serezha. This makes him lose.

2. Let ω be the circle in question, BL the tangent to this circle drawn from B, CC_1 the altitude of the triangle. We have $\angle HC_1A=90°$, that is, C_1 belongs to ω. It follows from the well-known property of secants that $BL^2=BC_1\cdot BA$.



The right triangles BAA_1 and BCC_1 are similar (since they have a common acute angle B), therefore, $\frac{BC_1}{BC}=\frac{BA_1}{BA}$, that is, $BC_1\cdot BA=BA_1\cdot BC$. The same argument shows that the right triangles BA_1D and BDC are similar, $\frac{BA_1}{BD}=\frac{BD}{BC}$, and $BA_1\cdot BC=BD^2$.

Combining these observations we get

$$BL^2=BC_1\cdot BA=BA_1\cdot BC=BD^2,$$

i.e. $BL=BD$.

3. See problem 2, senior league.

4. When all the variables are multiplied by $t>1$, the left side is multiplied by t^3, and the right one by t^2. Therefore it suffices to prove the inequality in the case $a^2+b^2+c^2=3$. Let $S=a+b+c$, then

$$ab+bc+ca=\frac{1}{2}(S^2-(a^2+b^2+c^2))=\frac{1}{2}(S^2-3)$$

so we can present the desired inequality in the form

$$S^3\geqslant\frac{9}{2}(S^2-3),$$

or, equivalently, $(S-3)^2(2S+3)\geqslant 0$. For positive S this inequality is obvious.